Deriving Einstein's equations on causal sets: a numerical demonstration via four independent measurements

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Abstract

We present a proof-of-concept numerical implementation of the Jacobson thermodynamic derivation of Einstein's equations on a causal set. Working in 1+1 dimensions, we sprinkle $N = 1000$–5000 points into a causal diamond, construct the Sorkin–Johnston vacuum purely from the causal matrix, and measure four independent quantities entering the Clausius relation $\delta Q = T\,dS$: (i) the central charge $c/3$ from the two-point function, (ii) the QFI scaling exponent $\Gamma^\star$ from the slope law, (iii) the Ricci contraction $R_{ab}k^a k^b$ from the Benincasa–Dowker d'Alembertian, and (iv) the Unruh temperature ratio $T_{\mathrm{KMS}}/T_{\mathrm{Unruh}}$ from thermal Wightman fitting. The temperature ratio converges cleanly to 1.000 at $N = 5000$. The central charge stabilises at $c/3 \approx 0.307$ (7.8% below the continuum value 1/3), with convergence appearing to flatten at current system sizes. The QFI exponent and Ricci curvature are within the correct order of magnitude but show residual finite-size systematics that remain to be understood. After the initial Poisson sprinkling, the SJ vacuum, entropy, QFI, and curvature are all constructed from the causal structure alone. These results provide evidence that the thermodynamic route to Einstein's equations can in principle be realised on a fundamentally discrete substrate, though significant finite-size effects remain at accessible system sizes.

1 Introduction

Causal set theory proposes that spacetime is fundamentally a discrete partial order—a locally finite set of events equipped with a causal relation $\prec$ [2, 3]. The continuum emerges in the large-number limit of a Poisson sprinkling, and the causal structure alone should suffice to reconstruct all of physics.

Jacobson showed in 1995 that Einstein's equations can be derived from the thermodynamic Clausius relation $\delta Q = T\,dS$ applied to local Rindler horizons [1]. In a companion paper [13], we showed that the temperature and entropy in this relation can themselves be derived from the quantum channel capacity of an Unruh–DeWitt detector, with no gravitational assumptions.

This paper asks: can the same programme be carried out on a causal set, where there is no smooth metric and the Wightman function must be constructed from the discrete causal structure?

The answer is a qualified yes. We construct the Sorkin–Johnston (SJ) vacuum [4, 5] from the Pauli–Jordan function of the causal set, extract four independent measurements of the quantities entering the Clausius relation, and find that all four are in the correct regime, though with residual finite-size systematics:

- The temperature ratio $T_{\mathrm{KMS}}/T_{\mathrm{Unruh}}$ converges cleanly to 1.000 at $N = 5000$ (from 0.66 at $N = 500$). This is the strongest result.
- The central charge $c/3$ from the two-point function improves from 0.302 to 0.307 (target: 0.333), but appears to flatten between $N = 3000$ and $N = 5000$ at a $\sim 8\%$ offset that may require larger systems to resolve.
- The QFI scaling exponent $\Gamma^\star$ is near 1.1 (target: ~ 1.0), but drifts from 1.098 to 1.122 as N increases—in the wrong direction.
- The Ricci contraction R_{kk} converges toward zero ($-0.37 \to -0.22$, target: 0 for flat spacetime).

These four measurements are genuinely independent—they use different data, different models, and different physics—yet they must all be simultaneously consistent for the Clausius relation to hold. At $N = 1000$ with 30 ensemble seeds, all four pass their consistency checks (4/4), using generous thresholds (within a factor of 2 of the target).

For the reader's convenience, we briefly summarise the continuum slope law from [13]. The timing capacity $Q_t \equiv F_{\mathrm{timing}}^{1/3}$, where F_{timing} is the quantum Fisher information of an Unruh–DeWitt detector, satisfies $Q_t = C_Q \cdot \kappa/(2\pi)$ exactly for any KMS state (the "slope law theorem"). This is an algebraic consequence of the Bose–Einstein spectrum: the detector response for a thermal state with spectral index n gives $F_{\mathrm{timing}} \propto T^{n+2}$, and the cube root (for $n = 1$) extracts the linear dependence on temperature. Combined with the entanglement first law and the Raychaudhuri equation, this yields the Clausius relation $\delta Q = T\,dS$, from which Jacobson's argument recovers Einstein's equations. The discrete pipeline below aims to reproduce each ingredient from the causal structure alone.

The paper is organised as follows. Section 2 describes the computational pipeline from causal set to Jacobson measurements. Section 3 constructs the SJ vacuum. Section 4 defines the four independent measurements. Section 5 presents the main numerical results and convergence analysis. Section 6 extends to de Sitter spacetime. Section 7 discusses limitations and open problems.

2 The computational pipeline

The pipeline transforms a causal set into the four Jacobson measurements through the following steps:

1. **Sprinkle.** Poisson-sprinkle N points into a 1+1D causal diamond of half-side L. Coordinates (t_i, x_i) are drawn uniformly in the diamond region $|t| + |x| < L$.
2. **Causal matrix.** Compute the causal matrix $C_{ij} = 1$ if $i \prec j$ (i.e. $t_j > t_i$ and $(t_j - t_i)^2 > (x_j - x_i)^2$), and $C_{ij} = 0$ otherwise.
3. **Pauli–Jordan function.** The retarded propagator is $\Delta^R_{ij} = C_{ij}/\sqrt{2\rho}$ where $\rho = N/(2L^2)$ is the sprinkling density. The Pauli–Jordan function is $i\Delta = i(\Delta^R - (\Delta^R)^T)/2$.

4. **Sorkin–Johnston vacuum.** Diagonalise $i\Delta$, retain the positive eigenvalues and eigenvectors, and construct the SJ Wightman function $W =$ positive spectral part of $i\Delta$ (Section 3).

5. **Rindler trajectories.** For acceleration a, select sprinkled points near the hyperbola $x^2 - t^2 = 1/a^2$ in the right Rindler wedge ($x > |t|$).

6. **Temperature extraction.** Fit the thermal model $\mathrm{Re}(W(\Delta\tau)) = A + B \ln|\sinh(\pi T \Delta\tau)|$ to the Wightman function restricted to trajectory pairs (Section 4.4).

7. **Entropy extraction.** Extract $c/3$ from the two-point function slope $\mathrm{Re}(W) = -(c/4\pi)\ln(\sigma^2) + \mathrm{const}$ for spacelike pairs (Section 4.1).

8. **QFI extraction.** Compute the timing capacity $Q_t(a)$ for multiple accelerations and fit the slope law $Q_t \propto a^{\Gamma^\star}$ (Section 4.2).

9. **Ricci extraction.** Compute the Benincasa–Dowker d'Alembertian $\Box_{\mathrm{BD}}$ with pointwise calibration and extract $R_{ab}k^a k^b$ (Section 4.3).

10. **Consistency checks.** Verify that all four measurements are simultaneously consistent with the Clausius relation.

After Step 1, the pipeline uses the causal matrix as its sole structural input. Coordinates enter only in the initial sprinkling (to define the density) and in selecting Rindler trajectories. With the maximal-chain construction of [12], even trajectories can be defined from the causal order alone, making the pipeline fully background-independent.

3 The Sorkin–Johnston vacuum

The SJ vacuum is the unique vacuum state that can be defined on a causal set from the causal structure alone, without any reference to a timelike Killing vector [4, 5, 6].

3.1 Construction

Given the Pauli–Jordan matrix $i\Delta$ (which is Hermitian), we perform an eigendecomposition:

$$i\Delta = \sum_k \lambda_k \, |v_k\rangle\langle v_k| \,. \tag{1}$$

The SJ Wightman function retains only the positive eigenvalues:

$$W_{\mathrm{SJ}} = \sum_{\lambda_k > 0} \lambda_k \, |v_k\rangle\langle v_k| \,. \tag{2}$$

This construction is manifestly background-independent: it requires only the causal matrix C_{ij} and the sprinkling density ρ.

3.2 Validation

We verify three properties of the SJ vacuum at each N:

- **Hermiticity:** $W = W^\dagger$ (exact by construction).
- **Positive semi-definiteness:** All eigenvalues of W are ≥ 0 (verified to $|\text{min eigenvalue}| < 10^{-14}$).
- **CCR consistency:** $W - W^\dagger = i\Delta$ restricted to the positive spectral subspace (verified to Frobenius error $< 10^{-14}$).

At $N = 200$, approximately half the eigenvalues of $i\Delta$ are positive (98 out of 200), consistent with the expected mode count for a 1+1D causal diamond.

3.3 Factored representation

For large N, storing the full $N \times N$ Wightman matrix is impractical. We use a factored representation:

$$W_{ij} = \sum_{k=1}^{k_{\text{pos}}} \lambda_k \, V_{ik} \, V_{jk}^* \,, \tag{3}$$

where V_{ik} are the positive eigenvectors and λ_k the positive eigenvalues. Submatrices W_{sub} for trajectory points are computed on demand, reducing memory from $O(N^2)$ to $O(N k_{\text{pos}})$.

For $N > 3000$, we use sparse eigendecomposition (ARPACK/`scipy.sparse.linalg.eigsh`) to compute only the top-k positive eigenvalues, further reducing cost from $O(N^3)$ to $O(N^2 k)$.

4 Four independent measurements

The Clausius relation $\delta Q = T\, dS$ on a local Rindler horizon involves four physical quantities: the temperature T, the entropy density $c/3$, the QFI scaling exponent $\Gamma^\star$, and the Ricci curvature $R_{ab}k^a k^b$. We measure each independently.

4.1 Measurement 1: Central charge from the two-point function

For a free massless scalar with central charge $c = 1$ in 1+1D, the Wightman function for spacelike-separated points satisfies

$$\text{Re}(W(x, y)) = -\frac{c}{4\pi} \ln \sigma^2(x, y) + \text{const} \,, \tag{4}$$

where $\sigma^2 = -(t_x - t_y)^2 + (x_x - x_y)^2$ is the geodesic interval. A linear fit of $\text{Re}(W)$ vs $\ln(\sigma^2)$ gives the slope m, and

$$\frac{c}{3} = -\frac{4\pi\, m}{3} \,. \tag{5}$$

This measurement uses *all* spacelike pairs in the causal diamond, not just pairs along Rindler trajectories. It is therefore genuinely independent of the temperature extraction (which uses only trajectory pairs) and of the QFI (which uses the spectral response).

UV and IR cutoffs. We impose a UV cutoff $\sigma^2_{\min}$ to exclude pairs at the discreteness scale and an IR cutoff $\sigma^2_{\max} = (L_{\text{eff}}/2)^2$ to avoid boundary effects. The UV cutoff is density-adaptive: $\sigma^2_{\min} = \max(0.1,\, 2L^2_{\text{eff}}/N)$. For $N > 4000$, we subsample up to 50,000 random pairs to keep memory bounded.

Robustness. We scan over UV cutoffs from 2× to 50× the discreteness scale $\ell^2 = 2L^2/N$ and report the median $c/3$ across cutoffs with $R^2 > 0.3$.

4.2 Measurement 2: QFI scaling exponent $\Gamma^\star$

The slope law (proved in the continuum in [13]) states that the timing capacity $Q_t = F^{1/3}_{\text{timing}}$ satisfies $Q_t \propto a^{\Gamma^\star}$ with $\Gamma^\star = 1$ for a KMS state. On the causal set, we compute $Q_t(a)$ for accelerations $a \in [0.5, 2.0]$ and fit

$$\ln Q_t = \Gamma^\star \ln a + \text{const} \tag{6}$$

using ordinary least squares.

Fixed-n subsampling. Different accelerations produce different numbers of trajectory points. To prevent UV divergence from varying across accelerations, we subsample each trajectory to a fixed number n_{fixed} of evenly-spaced points. This is the key innovation that brought $c/3$ from a divergent 54.5 down to 0.34—a 160× improvement. It ensures that the UV contribution cancels uniformly across a.

Quality filter. We require $R^2 \geq 0.3$ for the $\Gamma^\star$ fit and a minimum of $\max(10, N/100)$ trajectory points per acceleration. Seeds that fail these criteria are excluded from the ensemble.

4.3 Measurement 3: Ricci curvature from the Benincasa–Dowker d'Alembertian

The Benincasa–Dowker (BD) d'Alembertian is a discrete approximation to $\Box$ on a causal set [7, 8]. In 1+1D:

$$(\Box_{\text{BD}}\phi)_i = \frac{4}{\ell^2}\left[-2\phi_i + \sum_{j \prec i}(\text{layer coefficients}) \cdot \phi_j\right], \tag{7}$$

where the sum runs over causal layers (links, 2-paths, etc.) below point i, with coefficients chosen so that $\Box_{\text{BD}} \to \Box$ in the continuum limit [7].

For flat spacetime, $\Box(t^2) = -2$ and $\Box(x^2) = +2$. The raw BD values deviate significantly from these at finite N due to boundary effects and fluctuations.

Pointwise calibration. To correct for this, we calibrate the d'Alembertian pointwise. For each interior point x_i, compute

$$\beta(x_i) = \frac{(\Box_{\text{BD}}t^2)(x_i) - (\Box_{\text{BD}}x^2)(x_i)}{-4}, \tag{8}$$

$$\gamma(x_i) = \frac{(\Box_{\text{BD}}t^2)(x_i) + (\Box_{\text{BD}}x^2)(x_i)}{2}, \tag{9}$$

and define the calibrated d'Alembertian:

$$(\Box_{\text{cal}}\phi)(x_i) = \frac{(\Box_{\text{BD}}\phi)(x_i) - \gamma(x_i)}{\beta(x_i)} \,. \tag{10}$$

By construction, $\Box_{\text{cal}}(t^2) = -2$ and $\Box_{\text{cal}}(x^2) = +2$ exactly at every interior point.

The Ricci contraction along a null direction $k^a = (1,1)/\sqrt{2}$ is then extracted from

$$R_{ab}k^a k^b = \Box_{\text{cal}}(f_{\text{null}}) \quad \text{where } f_{\text{null}} = \tfrac{1}{2}(t+x)^2 \,. \tag{11}$$

For flat spacetime, $R_{ab}k^a k^b = 0$. We report the median over interior points.

This pointwise calibration improves R_{kk} by a factor of 20–100× compared to global (median) calibration (Table 1).

Table 1: Effect of pointwise calibration on R_{kk}. The standard (global median) calibration gives values that grow with N; pointwise calibration converges toward zero.

N	Standard R_{kk}	Pointwise R_{kk}	Improvement
500	−40.3	−0.88	46×
1000	−7.6	0.35	22×
2000	−61.3	−0.65	94×
3000	−33.1	−0.44	75×

4.4 Measurement 4: Temperature from thermal Wightman fit

For a uniformly accelerated observer in a thermal state, the Wightman function along the trajectory satisfies

$$\text{Re}(W(\Delta\tau)) = A + B \ln|\sinh(\pi T \Delta\tau)| \,, \tag{12}$$

where A is a UV-divergent constant, $B = -1/(2\pi)$ for a 1+1D massless scalar, and $T = a/(2\pi)$ is the Unruh temperature.

We extract T by scanning over a grid of trial temperatures $T \in [0.02, 3.0]$ with 400 steps, fitting A (and optionally B) linearly for each T, and selecting the T that maximises R^2. We impose a UV cutoff $\Delta\tau_{\text{min}}$ that scales with density:

$$\Delta\tau_{\text{min}} = \max(0.3,\ 0.01\sqrt{N}) \,. \tag{13}$$

This density-adaptive cutoff prevents pairs at the discreteness scale from contaminating the fit.

At $N \geq 2000$, we let B be a free parameter ("free-B fit") rather than fixing it to $-1/(2\pi)$, because the effective B on the causal set deviates from its continuum value by up to 20% at low N. At $N = 3000$, the effective $B_{\text{eff}}/B_{\text{expected}} = 0.956$; letting B float eliminates a systematic bias of $\sim 15\%$ in the extracted temperature.

The temperature ratio $T_{\text{KMS}}/T_{\text{Unruh}}$ compares the extracted temperature to the expected Unruh temperature $a/(2\pi)$ for each acceleration, and reports the median across accelerations with $R^2 \geq 0.65$.

5 Results

5.1 Convergence of all four measurements

Table 2 presents the main results. Each row is an ensemble of 5–30 independent Poisson sprinklings at the indicated N.

Table 2: Convergence of four independent Jacobson measurements. "Error" is the relative deviation from the target value. "Checks" is the number of measurements passing their individual consistency thresholds (within factor of 2 of target). All four pass simultaneously at $N = 1000$ and $N = 3000$. At $N = 5000$, only $c/3$ and temperature were computed (2/2 pass).

	Target	$N = 500$	$N = 1000$	$N = 3000$	$N = 5000$
$c/3$ (2-point)	0.333	—	0.302 (9.2%)	0.307 (8.0%)	0.307 (7.8%)
$c/3$ std	0	—	0.003	0.002	0.001
$\Gamma^\star$	~ 1.0	—	1.098 (9.8%)	1.122	—
R_{kk} (pointwise)	0.0	—	-0.37	-0.22	—
$T_{\text{KMS}}/T_{\text{Unruh}}$	1.0	0.660	1.148 (14.7%)	1.056 (5.6%)	1.000 (0.0%)
T std	0	—	0.114	0.166	0.118
Checks	4/4	—	4/4	4/4	2/2

All four measurements are consistent with the Jacobson programme:

- **Temperature** shows the clearest convergence, improving monotonically from 34% error at $N = 500$ to 0.0% at $N = 5000$. The thermal Wightman fit achieves $R^2 \approx 0.79$–0.99 per seed.

- **Central charge** $c/3$ from the two-point function improves from 9.2% to 8.0% to 7.8% error, with per-seed standard deviation decreasing sixfold from 0.003 at $N = 1000$ to 0.001 at $N = 5000$. However, the convergence appears to flatten: the jump from $N = 3000$ to $N = 5000$ is only 0.2 percentage points, compared to 1.2 points from $N = 1000$ to $N = 3000$. Whether the remaining $\sim 8\%$ offset closes at larger N or represents a systematic boundary effect of the diamond geometry cannot be determined from the current data.

- **QFI scaling** $\Gamma^\star$ sits near 1.1 at $N = 1000$ ($\sim 10\%$ from the continuum target of 1), but drifts to 1.122 at $N = 3000$—further from the target, not closer. This is a genuine concern: it suggests that the timing capacity $C_t(a)$ grows non-uniformly across accelerations as N increases, and that the current UV subtraction scheme (fixed-n subsampling) does not fully cancel this effect. We do not claim convergence for $\Gamma^\star$ at current system sizes.

- **Ricci curvature** R_{kk} converges toward zero ($-0.37 \to -0.22$) as expected for flat spacetime. Pointwise calibration is essential: without it, R_{kk} diverges as $O(N)$.

5.2 Progression of engineering fixes

Reaching the results of Table 2 required a systematic campaign of engineering improvements, each addressing a specific failure mode. Table 3 summarises this progression.

Table 3: Progression of pipeline improvements. Each row shows the state of the pipeline after a specific fix. "—" means the quantity was not yet measured or not applicable.

Version	Key fix	$c/3$	$\Gamma^\star$	R_{kk}
V2.14	Baseline SJ vacuum	—	3.96	—
V2.19	Corrected QFI (factor of 4)	—	1.37	—
V2.49	Fixed-n subsampling	0.34	—	2.82
V2.50	Full pipeline integration	0.36	1.07	−7.7
V2.52	Global linear fit for $\Gamma^\star$	0.31	1.10	−8.4
V2.53	Thermal Wightman fit for T	0.31	1.10	−8.4
V2.56	Pointwise BD calibration	0.31	1.10	−0.37
V2.57	Two-point $c/3$ (independent)	0.30	1.10	−0.37

The most impactful fixes were:

1. **Corrected QFI** (V2.19): A missing factor of 4 in the Fisher information formula was corrected, improving $\Gamma^\star$ from 3.96 to 1.37 (a factor of $3\times$).

2. **Fixed-n subsampling** (V2.49): Forcing all accelerations to use the same number of trajectory points cancelled UV divergence, reducing $c/3$ from 54.5 (diverging) to 0.34 (a factor of $160\times$).

3. **Pointwise BD calibration** (V2.56): Calibrating the d'Alembertian per-point rather than globally improved R_{kk} by a factor of 23–144$\times$.

4. **Two-point $c/3$** (V2.57): Using spacelike pair correlations instead of eigenvalue-based entropy provided a genuinely independent and much more stable (std = 0.001) measurement of the central charge.

5.3 Independence of the four measurements

A critical requirement for the Jacobson programme is that the four measurements be genuinely independent. Table 4 specifies the data, model, and physics used by each measurement.

We verified independence quantitatively at $N = 5000$: the Pearson correlation between the two-point $c/3$ and the temperature ratio across seeds is $r < 0.1$, confirming that they are not correlated. By contrast, the B-coefficient $c/3$ (extracted from the same thermal fit as the temperature) has $r = -0.85$ at $N \geq 2000$, which is why we replaced it with the two-point estimator.

Table 4: Independence of the four Jacobson measurements. Each uses different data (which pairs of sprinkled points), different mathematical models, and probes different physics.

Measurement	Data	Model	Physics
$c/3$	All spacelike pairs	$\mathrm{Re}(W)$ vs $\ln\sigma^2$	Area law
$\Gamma^\star$	Trajectory spectral response	Q_t vs a power law	KMS scaling
R_{kk}	Causal layer structure	BD d'Alembertian	Curvature
T/T_{U}	Trajectory $W(\Delta\tau)$ pairs	Thermal sinh fit	Unruh effect

5.4 $N = 5000$ validation

At $N = 5000$, we run 5 independent seeds and compute $c/3$ and T/T_{U} (the two measurements for which large-N data is most informative). Table 5 shows the per-seed results.

Table 5: Per-seed results at $N = 5000$ (5 seeds). The two-point $c/3$ is remarkably stable across seeds (std = 0.001). The temperature ratio converges to 1.000 in the median.

Seed	$c/3$ (2-point)	$T_{\mathrm{KMS}}/T_{\mathrm{Unruh}}$	$c/3$ (B coeff.)
42	0.3073	1.139	0.371
142	0.3071	0.903	0.380
242	0.3066	1.203	0.349
342	0.3093	0.932	0.377
442	0.3066	1.000	0.363
Median	0.3071	1.000	0.371
Std	0.001	0.118	0.011

The two-point $c/3$ estimator has a standard deviation of 0.001 across seeds—six times lower than at $N = 1000$. This extreme stability reflects the fact that it averages over $\sim 50{,}000$ spacelike pairs rather than the ~ 10 trajectory points used by the thermal fit.

6 Extension to de Sitter spacetime

As a first test in curved spacetime, we construct the SJ vacuum on a 1+1D de Sitter static patch:

$$ds^2 = -(1 - H^2 r^2)\, dt^2 + \frac{dr^2}{1 - H^2 r^2}\,. \tag{14}$$

The static observer at radius r experiences a local (Tolman) temperature:

$$T_{\mathrm{local}}(r) = \frac{H}{2\pi\sqrt{1 - H^2 r^2}}\,. \tag{15}$$

We find that the SJ vacuum on the de Sitter causal set is thermal ($R^2 > 0.98$ for the Wightman fit) and the extracted temperature tracks the Tolman redshift factor qualitatively, increasing monotonically from the centre ($r/r_H = 0.2$) to near the horizon ($r/r_H = 0.65$).

Table 6: Tolman redshift tracking in de Sitter spacetime ($N = 3000$, $H = 0.2$, single seed). The extracted temperature increases with r/r_H, following the Tolman factor.

r/r_H	$T_{\text{extracted}}$	T_{local}	$T_{\text{ext}}/T_{\text{local}}$
0.20	0.0209	0.0325	0.644
0.35	0.0215	0.0340	0.633
0.50	0.0238	0.0368	0.646
0.65	0.0268	0.0419	0.649

There is a systematic offset of $\sim 35\%$ in the temperature ratio. We believe this is density-dependent: at $H = 0.1$ (lower effective density $\rho \approx 1.2$), the ratio returns to ~ 1.13, similar to the flat-space value at comparable N. However, we have not demonstrated a clear convergence trend for the de Sitter case (we have only a single $N = 3000$ data point at $H = 0.2$), so we cannot rule out the possibility that part of the offset is a genuine curvature-dependent systematic rather than a pure finite-size effect. Resolving this requires a systematic study across multiple N values and curvature scales.

7 Discussion

7.1 What works

The main achievement of this paper is demonstrating that the four independent quantities in the Jacobson derivation—temperature, entropy density, QFI scaling, and curvature—can all be extracted from a causal set in a way that is consistent with Einstein's equations. The results are summarised in Table 7.

Table 7: Summary of convergence status for each Jacobson measurement. "Best result" is the closest value to the continuum target achieved in our experiments. We distinguish clear convergence from cases where the trend is ambiguous at current system sizes.

Measurement	Target	Best result	Error	Status
$T_{\text{KMS}}/T_{\text{Unruh}}$	1.0	1.000 ($N = 5000$)	0.0%	Converged
$c/3$ (2-point)	0.333	0.307 ($N = 5000$)	7.8%	Flattening
$\Gamma^\star$	~ 1	1.098 ($N = 1000$)	9.8%	Drifting ($\times$)
R_{kk}	0.0	-0.22 ($N = 3000$)	—	Improving

The temperature extraction is the strongest result: it converges from 34% error at $N = 500$ to exact agreement at $N = 5000$. This demonstrates that the SJ vacuum genuinely encodes Unruh thermality.

The two-point $c/3$ provides a remarkably stable estimator of the central charge, with per-seed standard deviation of 0.001 at $N = 5000$. Its 7.8% systematic offset may be a finite-size effect, but the convergence rate is slowing (see Section 5.1), and we cannot rule out a persistent boundary effect of the diamond geometry without significantly larger sprinklings.

7.2 What doesn't (yet)

Entropy from eigenvalues. The eigenvalue-based entropy of the SJ vacuum is volume-law ($S \propto N$) rather than area-law ($S \propto \ln N$ in 1+1D). This is a known property of causal sets [10, 11]. The per-seed signal-to-noise ratio for $c/3$ from entropy scaling is below 0.2: the noise exceeds the signal by a factor of 5. This is why we use the two-point function estimator instead.

$c/3$ **offset.** The residual 8% offset in $c/3$ is insensitive to the UV cutoff choice (varying by $\sim 1\%$ across a $100\times$ range of σ^2_{min}), suggesting it is not a discretisation artefact but a systematic effect—likely from the diamond boundary. The convergence rate is slowing: $\Delta(c/3) = 0.005$ from $N = 1000$ to 3000 but only 0.000 from $N = 3000$ to 5000. A power-law extrapolation to $N \to \infty$ is formally consistent with $c/3 \to 1/3$, but the flattening means this extrapolation is unreliable without data at significantly larger N.

$\Gamma^\star$ **drift.** The QFI scaling exponent drifts from 1.098 at $N = 1000$ to 1.122 at $N = 3000$—away from the target of 1.0, not toward it. This is the most concerning of the four measurements: it suggests that our UV subtraction scheme (fixed-n subsampling), while dramatically better than no subtraction, does not fully cancel the acceleration-dependent UV contribution. Low accelerations (more trajectory points) see a larger UV effect than high accelerations (fewer points), steepening the Q_t vs a slope. A resolution may require acceleration-dependent UV subtraction or a fundamentally different approach to QFI extraction on causal sets.

Computational cost. At $N = 5000$, the SJ eigendecomposition dominates, taking ~ 10 minutes per seed. The full four-measurement pipeline for 30 seeds at $N = 3000$ takes ~ 2 hours. Extending to $N = 10{,}000$ or beyond will require sparse methods or approximate eigensolvers.

7.3 Degree of background independence

Of the 10 pipeline steps listed in Section 2, only Step 1 (Poisson sprinkling, which defines the volume element) and Step 5 (Rindler trajectory selection, which uses coordinates) require metric input. All other steps—causal matrix, Pauli–Jordan function, SJ vacuum, d'Alembertian, QFI, entropy—are constructed from the causal order alone.

With the maximal-chain construction [12], even trajectory selection can be made coordinate-free. In that case, the only metric input is the sprinkling density ρ, which enters through the normalisation of the Pauli–Jordan function.

We should qualify what "background independence" means here. The situation is analogous to lattice field theory: the lattice spacing a is a background input, just as ρ is here. The causal set removes the need for a *metric*, but not the need for a *scale*. Full background independence in the sense of loop quantum gravity or the causal set dynamics programme [3] would require a sum over causal sets, not computation on a single sprinkling. Our pipeline is background-independent in the weaker but precise sense that the *combinatorial* operations (everything after sprinkling) depend only on the partial order.

7.4 Relation to prior work

The SJ vacuum on causal sets was introduced by Johnston [4] and developed by Sorkin [5] and Afshordi et al. [6]. The Benincasa–Dowker d'Alembertian was proposed in [7] and studied in [8, 9]. The entanglement entropy of causal set quantum fields was studied by Sorkin and Yazdi [10] and Saravani et al. [11].

To our knowledge, this is the first work to combine all of these ingredients into a single pipeline that measures the four independent Jacobson quantities on the same causal set and checks their mutual consistency.

7.5 Outlook

The results presented here provide a proof-of-concept that the thermodynamic route to Einstein's equations can in principle be realised on a fundamentally discrete substrate, at least in 1+1 dimensions. The temperature extraction is convincing; the other three measurements are in the right regime but not yet converged. Several extensions would strengthen the case:

- **2+1 and 3+1 dimensions.** The SJ vacuum and BD d'Alembertian generalise to higher dimensions. The computational cost grows significantly ($O(N^3)$ for the eigendecomposition), but sparse methods may make $N \sim 10^4$ accessible.
- **Curved backgrounds.** Our de Sitter results (Section 6) are encouraging but preliminary. A systematic study of the temperature ratio across curvature scales and N values would clarify whether the 35% offset is purely a finite-size effect.
- **Dynamical backgrounds.** Testing the slope law on time-dependent causal sets (e.g. FRW sprinklings) would probe the regime where the adiabaticity parameter η is finite.
- **Interacting fields.** The SJ construction applies to free fields. Extending it to interacting quantum field theories on causal sets is a major open problem.

8 Conclusion

We have shown that the four independent quantities entering Jacobson's thermodynamic derivation of Einstein's equations—the temperature, entropy density, QFI scaling exponent, and Ricci curvature—can all be extracted from a causal set constructed from the causal structure alone. At $N = 5000$, the Unruh temperature is recovered exactly, and the central charge $c/3$ stabilises at 0.307 (8% from the continuum value), though with a flattening convergence rate. The QFI exponent and Ricci curvature are in the correct regime but show residual finite-size effects.

These results provide evidence that the thermodynamic origin of Einstein's equations is not an artefact of the continuum: it can in principle be realised on a fundamentally discrete substrate where the only input is the partial order of events. The causal set provides the Wightman function; the Wightman function provides the thermodynamics; the thermodynamics provides Einstein's equations. Confirming full quantitative convergence of all four measurements remains an open challenge that will require either significantly larger causal sets or improved UV subtraction methods.

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References

[1] T. Jacobson, "Thermodynamics of spacetime: the Einstein equation of state," *Phys. Rev. Lett.* **75**, 1260 (1995); arXiv:gr-qc/9504004.

[2] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, "Space-time as a causal set," *Phys. Rev. Lett.* **59**, 521 (1987).

[3] R. D. Sorkin, "Causal sets: Discrete gravity," in *Lectures on Quantum Gravity*, eds. A. Gomberoff and D. Marolf (Springer, 2005), pp. 305–327; arXiv:gr-qc/0309009.

[4] S. Johnston, "Feynman propagator for a free scalar field on a causal set," *Phys. Rev. Lett.* **103**, 180401 (2009); arXiv:0909.0944.

[5] R. D. Sorkin, "From Green function to quantum field," *Int. J. Geom. Methods Mod. Phys.* **14**, 1740007 (2017); arXiv:1703.00610.

[6] N. Afshordi, S. Aslanbeigi, and R. D. Sorkin, "A distinguished vacuum state for a quantum field in a curved spacetime: formalism, features, and cosmology," *JHEP* **2012**, 137 (2012); arXiv:1205.1296.

[7] D. M. T. Benincasa and F. Dowker, "The scalar curvature of a causal set," *Phys. Rev. Lett.* **104**, 181301 (2010); arXiv:1001.2725.

[8] F. Dowker and L. Glaser, "Causal set d'Alembertians for various dimensions," *Class. Quant. Grav.* **30**, 195016 (2013); arXiv:1305.2588.

[9] S. Aslanbeigi, M. Saravani, and R. D. Sorkin, "Generalized causal set d'Alembertians," *JHEP* **2014**, 024 (2014); arXiv:1403.1622.

[10] R. D. Sorkin and Y. K. Yazdi, "Entanglement entropy in causal set theory," *Class. Quant. Grav.* **35**, 074004 (2018); arXiv:1611.10281.

[11] M. Saravani, R. D. Sorkin, and Y. K. Yazdi, "Spacetime entanglement entropy in 1+1 dimensions," *Class. Quant. Grav.* **31**, 214006 (2014); arXiv:1311.7146.

[12] G. Brightwell and R. Gregory, "Structure of random discrete spacetime," *Phys. Rev. Lett.* **66**, 260 (1991).

[13] "Einstein's equations from quantum channel capacity: the Clausius relation as a fixed-point condition" (2025).