Einstein's equations from quantum channel capacity: the Clausius relation as a fixed-point condition

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Abstract

We identify a quantum-information observable—the timing capacity $Q_t \equiv F_{\mathrm{timing}}^{1/3}$, constructed from the quantum Fisher information of an Unruh–DeWitt detector—that encodes horizon temperature in a particularly clean way. We prove the *slope law theorem*: for any KMS state, $Q_t = C_Q\, \kappa/(2\pi)$ exactly, where C_Q depends only on the detector switching function. This is an algebraic consequence of the KMS condition via the Bose–Einstein spectrum, holding with zero free parameters. We recast the entropy density as a function of Q_t and combine this with the entanglement first law to obtain the Clausius relation $\delta Q = T\, dS$, then feed these into Jacobson's 1995 argument to recover $G_{ab} + \Lambda\, g_{ab} = 8\pi G\, T_{ab}$. The gravitational content enters through the Wightman function, which encodes the background geometry; the field equations emerge as a consistency condition that the background must satisfy for the Clausius relation to hold on all local horizons. The main result is a *fixed-point characterisation* of GR: we show that the capacity-derived Clausius residual vanishes if and only if the background satisfies Einstein's equations, and grows linearly with any deformation—$f(R)$ gravity, wrong coupling constant, wrong equation of state, and Brans–Dicke theory all produce nonzero residuals. General relativity is the unique fixed point.

1 Introduction

In 1995, Jacobson showed that Einstein's field equations follow from three ingredients: the proportionality of entropy and horizon area, the Clausius relation $\delta Q = T\, dS$, and the Raychaudhuri equation [1]. The argument is elegant, but it takes the temperature $T = \kappa/(2\pi)$ and the entropy–area law $S = A/(4G)$ as phenomenological inputs. One would like to express these thermodynamic quantities in terms of concrete quantum-information observables.

This paper does three things. First, we identify a specific observable—the timing capacity Q_t of an Unruh–DeWitt (UDW) detector—that encodes horizon temperature and entropy density in a particularly clean way, and prove the *slope law theorem*: $Q_t \propto \kappa$ exactly for any KMS state (Theorem 3). Second, we recast the Clausius relation in terms of Q_t and carry out Jacobson's argument. Third, and most importantly, we show that the resulting Clausius relation provides a sharp *fixed-point characterisation* of general relativity: GR is the unique metric theory for which the residual vanishes, and any deformation produces a linearly growing violation.

We are careful about what is genuinely new here. The slope law theorem is an algebraic consequence of the KMS condition; it repackages the standard result that thermal detectors respond with a Bose–Einstein spectrum into a clean scaling relation for a specific QFI observable. The entropy–capacity connection (Section 3) is a rewriting of the Stefan–Boltzmann law in capacity language. The Jacobson derivation itself (Section 4) is standard [1].

The non-trivial content lies in two places. First, the slope law identifies a *specific* quantum-information quantity ($F_{\text{timing}}^{1/3}$) that linearises the temperature dependence with zero free parameters and holds universally across detector families—this is a concrete bridge between quantum information and horizon thermodynamics that, as far as we know, has not been exhibited before. Second, the non-GR tests of Section 5 go beyond the original Jacobson argument: we show not only that GR satisfies the Clausius relation, but that every tested alternative fails it, with the residual growing linearly in the deformation parameter.

A word on the role of the Wightman function. The entire gravitational content of the background enters through $G^+(\Delta\tau)$. One might object that this makes the derivation circular: the answer enters through the input. This is partly correct. The point is not that we derive gravity from nothing, but that we identify a *consistency condition*—the vanishing of the Clausius residual on all local horizons—that the background must satisfy, and that this condition singles out Einstein's equations uniquely. The chain of logic is:

$$\text{Wightman } G^+(\Delta\tau) \xrightarrow{\text{QFI}} Q_t \xrightarrow{\text{slope law}} T \xrightarrow{\text{entropy}} S \xrightarrow{\text{Clausius + Raychaudhuri}} G_{ab}+\Lambda\, g_{ab} = 8\pi G\, T_{ab}\,. \tag{1}$$

The paper is organised as follows. Section 2 defines the timing capacity and proves the slope law theorem. Section 3 establishes the entropy–capacity connection and derives the Clausius relation from the entanglement first law. Section 4 carries out Jacobson's argument with capacity-derived inputs. Section 5 presents the non-GR tests—the main result of the paper. Section 6 extends the slope law beyond strict equilibrium. Section 7 discusses the results, their limitations, and open problems.

2 Timing capacity and the slope law

2.1 Setup

Consider a pointlike Unruh–DeWitt detector coupled linearly to a real scalar field ϕ in $d{+}1$ dimensions. The detector follows a worldline $x^\mu(\tau)$ with proper time τ and interacts via the Hamiltonian $H_{\text{int}} = \lambda\, \chi(\tau)\, \mu(\tau)\, \phi(x(\tau))$, where $\chi(\tau)$ is a switching function and $\mu(\tau)$ is the detector's monopole operator.

For a Gaussian switching function $\chi(\tau) = \exp(-\tau^2/(2\sigma^2))$ with width σ, the transition rate of the detector to first order in λ is [2, 3]

$$\dot{\mathcal{F}}(\Omega) = \int_{-\infty}^{\infty} d(\Delta\tau)\; e^{-i\Omega\Delta\tau}\, e^{-\Delta\tau^2/(4\sigma^2)}\, G^+(\Delta\tau)\,, \tag{2}$$

where $G^+(\Delta\tau) = \langle 0|\phi(x(\tau))\phi(x(\tau'))|0\rangle$ is the Wightman two-point function along the trajectory and Ω is the detector energy gap.

Definition 1 (Timing Fisher information)**.** The quantum Fisher information for estimating the surface gravity κ from the detector response is

$$F_{\text{timing}} = \max_{\Omega}\left[4\,\Omega^2\,|\dot{\mathcal{F}}(\Omega)|\right]. \tag{3}$$

Definition 2 (Timing capacity)**.**

$$Q_t \equiv F_{\mathrm{timing}}^{1/3} . \tag{4}$$

The exponent 1/3 is not arbitrary. It is uniquely determined by requiring that $Q_t \propto \kappa$ (see Theorem 3 below).

2.2 The slope law theorem

Theorem 3 (Slope law for KMS states)**.** *Let $G^+(\Delta\tau)$ be the Wightman function of a KMS state at temperature $T = \kappa/(2\pi)$. Then*

$$Q_t = C_Q \cdot T = C_Q \cdot \frac{\kappa}{2\pi} , \tag{5}$$

where $C_Q = (4A\, c_3)^{1/3}$ depends only on the switching function parameter σ through $A = \sqrt{\pi}\,\sigma/(2\pi)$, and $c_3 = x_\star^3/(e^{x_\star} - 1) \approx 1.4214$ with $x_\star \approx 2.8214$ the unique positive solution of $3(1 - e^{-x}) = x$.

Proof. The proof proceeds in five steps. Each step is an algebraic identity for KMS states; we verify each numerically to confirm correctness of our implementation.

Step 1: KMS condition implies detailed balance. The KMS condition $G^+(\Delta\tau + i\beta) = G^+(\Delta\tau)$ implies, after Fourier transform, the detailed balance relation

$$\dot{\mathcal{F}}(-\Omega)/\dot{\mathcal{F}}(+\Omega) = e^{-\Omega/T} . \tag{6}$$

Numerically: the maximum deviation from (6) is below 10^{-15} for all temperatures tested ($T = 0.05$–5.0).

Step 2: Detailed balance implies Bose–Einstein spectrum. For a UDW detector (spectral index $n = 1$), the response takes the form

$$\dot{\mathcal{F}}(\Omega) = \frac{A\,|\Omega|}{e^{|\Omega|/T} - 1} \quad \text{for } \Omega > 0 , \tag{7}$$

where $A = \sqrt{\pi}\,\sigma/(2\pi)$ is a normalisation constant set by the switching function.

Step 3: Bose–Einstein spectrum determines F_{timing}. Substituting (7) into (3):

$$F_{\mathrm{timing}} = \max_{\Omega}\left[\frac{4A\,\Omega^3}{e^{\Omega/T} - 1}\right] = 4A\,T^3 \max_{x}\left[\frac{x^3}{e^x - 1}\right] = 4A\, c_3\, T^3 , \tag{8}$$

where $x = \Omega/T$ and $c_3 = x_\star^3/(e^{x_\star} - 1)$ with $x_\star$ satisfying $3 - 3e^{-x_\star} = x_\star$ (i.e. $x_\star \approx 2.8214$, $c_3 \approx 1.4214$).

Step 4: Cube root extracts linear scaling. Taking the cube root:

$$Q_t = F_{\mathrm{timing}}^{1/3} = (4A\, c_3)^{1/3}\, T = C_Q\, T . \tag{9}$$

Step 5: Linearity in κ. Since $T = \kappa/(2\pi)$, a power-law fit of Q_t vs κ yields $Q_t \propto \kappa^\alpha$ with $\alpha = 1.000000$ and residual $< 10^{-15}$. □

Table 1 shows that C_Q depends on σ but the exponent $\alpha = 1$ is universal.

Table 1: Slope law verification for different switching widths σ. The power-law exponent α (in $Q_t \propto \kappa^\alpha$) is unity to machine precision for all σ, while C_Q varies as predicted by $(4A\,c_3)^{1/3}$.

σ	C_Q (predicted)	C_Q (measured)	α	Residual
1.0	1.171	1.170	1.0000	9×10^{-16}
2.0	1.475	1.475	1.0000	5×10^{-16}
4.0	1.858	1.858	1.0000	6×10^{-16}
8.0	2.341	2.341	1.0000	6×10^{-16}
16.0	2.950	2.950	1.0000	4×10^{-16}
32.0	3.716	3.715	1.0000	5×10^{-16}

2.3 Why the exponent 1/3?

The UDW detector has spectral index $n = 1$ (the response scales as $|\Omega|$ times the Bose–Einstein factor). For a general detector with spectral index n, $\dot{\mathcal{F}} \propto |\Omega|^n/(e^{|\Omega|/T} - 1)$, and the timing QFI scales as

$$F_{\text{timing}} \propto T^{n+2} \,. \tag{10}$$

The slope law $Q_t \propto \kappa$ then requires $Q_t = F_{\text{timing}}^{1/(n+2)}$. For UDW ($n = 1$), this gives $1/(1+2) = 1/3$. We have verified this numerically for $n = 0, 1, 2, 3$ (amplitude, UDW, derivative, and quadratic coupling) with all giving $\alpha = 1.0000$ and residuals below 10^{-15}.

This universality extends across detector families: any detector whose response can be written as $|\Omega|^n \cdot n_{\text{BE}}(\Omega/T)$ admits a choice of Q_t for which the slope law holds. The slope law is not a property of a particular detector—it is a property of the KMS condition.

2.4 KMS condition as necessary condition

The converse also holds: if the state is not KMS, the slope law fails. We test this by adding a fixed-frequency perturbation of amplitude ε to the thermal spectrum. Table 2 shows the result.

Table 2: Slope law violation for non-KMS states. A fixed-frequency perturbation of amplitude ε is added to the thermal spectrum. The slope law holds if and only if $\varepsilon = 0$ (exact KMS).

ε	α	$\lvert\Gamma^\star - 1\rvert$
0.000	1.000	6×10^{-6}
0.001	0.997	3.3×10^{-3}
0.010	0.875	1.4×10^{-1}
0.100	0.395	1.5
0.500	0.133	6.5

3 Entropy from capacity and the Clausius relation

3.1 Per-mode entropy from detector response

The detector response (7) determines the occupation number of each mode:

$$\bar{n}(\omega) = \frac{\dot{\mathcal{F}}(\omega)}{A\,\omega}\,, \tag{11}$$

and the von Neumann entropy per mode is the Bose–Einstein entropy function:

$$s(\bar{n}) = (1+\bar{n})\ln(1+\bar{n}) - \bar{n}\ln\bar{n}\,. \tag{12}$$

This is exact and non-circular: the detector response alone determines the entropy, with no reference to temperature.

3.2 Entropy scaling laws

Integrating over modes in 1+1 dimensions:

$$\frac{S}{L} = \frac{1}{2\pi}\int_0^\infty s\left(\frac{1}{e^{\omega/T}-1}\right)d\omega = \frac{T}{2\pi}\int_0^\infty s\left(\frac{1}{e^{x}-1}\right)dx = \frac{\pi T}{6}\,, \tag{13}$$

where the integral evaluates to $\pi^2/3$ (a standard result). Substituting $T = Q_t/C_Q$ from the slope law:

$$\frac{S}{L} = \frac{\pi}{6\,C_Q}\,Q_t\,. \tag{14}$$

The entropy density is a *linear function* of the timing capacity. In 3+1 dimensions, the corresponding result is cubic: $s/V = (2\pi^2/(45\,C_Q^3))\,Q_t^3$.

Table 3 verifies (14) numerically. The power-law fit gives $S \propto Q_t^{1.0000}$ with residual below 10^{-15}.

Table 3: Entropy–capacity relation in 1+1D. The numerical entropy $S_{\rm num}$ agrees with the capacity prediction $S_{\rm cap} = (\pi/(6C_Q))Q_tL$ to four decimal places.

κ	T	Q_t	$S_{\rm num}/L$	$S_{\rm cap}/L$
0.50	0.080	0.186	0.04167	0.04166
3.00	0.477	1.118	0.25003	0.24994
5.50	0.875	2.049	0.45839	0.45823
8.00	1.273	2.980	0.66675	0.66651

3.3 Universality across gravity theories

A striking feature of the entropy–capacity relation is that it holds unchanged in modified gravity. We test this on $f(R) = R + \alpha R^2$ backgrounds. In this theory, the physical (Wald) temperature is $T_{\rm Wald} = H/(2\pi f'(R))$ rather than the geometric $T_{\rm GR} = H/(2\pi)$. The capacity-derived temperature automatically picks up the $f'(R)$ correction (as shown in Section 5.6), and the entropy–capacity relation $S = f(Q_t)$ continues to hold with the same function f:

This is important: the entropy function depends on the field content (the scalar field), not on the gravitational dynamics. The gravity theory affects Q_t (through the Wightman function), but the map $Q_t \mapsto S$ is the same.

Table 4: Entropy–capacity relation on $f(R) = R + \alpha R^2$ de Sitter backgrounds. The function $f(Q_t)$ is universal across gravity theories.

α	T_{Wald}	S_{Wald}/L	S_{cap}/L	Match
0.000	0.1592	0.0833	0.0833	Yes
0.001	0.1554	0.0814	0.0814	Yes
0.010	0.1284	0.0672	0.0672	Yes
0.050	0.0723	0.0379	0.0379	Yes

3.4 The Clausius relation from the entanglement first law

The entanglement first law states that for a perturbation of a thermal state,

$$\delta\langle K\rangle = \delta S\,, \tag{15}$$

where $K = -\ln\rho_0$ is the modular Hamiltonian. This is an exact identity of quantum mechanics [4, 5], not an approximation.

For a thermal state with modular Hamiltonian $K = \sum_\omega (\omega/T)\,\hat{n}_\omega$, the mathematical identity underlying this is:

$$\frac{ds}{d\bar{n}} = \ln\left(1 + \frac{1}{\bar{n}}\right) = \ln(e^{\omega/T}) = \frac{\omega}{T}\,, \tag{16}$$

which holds exactly for the Bose–Einstein distribution $\bar{n} = 1/(e^{\omega/T} - 1)$. This is the fundamental identity that connects entropy to temperature without any gravitational input.

The Clausius relation follows immediately. Define the heat as the change in the reduced Hamiltonian:

$$\delta Q \equiv \delta\langle H_R\rangle = T\,\delta\langle K\rangle = T\,\delta S\,. \tag{17}$$

Every quantity in this chain has been derived from the timing capacity: T from the slope law, S from the entropy–capacity relation, and δQ from the entanglement first law.

We verify numerically that $\delta\langle K\rangle = \delta S$ to machine precision ($|\delta\langle K\rangle - \delta S|/|\delta S| < 10^{-13}$) across temperatures $T = 0.1$–5.0, and that the Clausius relation $\delta Q = T\,dS$ reproduces the correct Stefan–Boltzmann specific heat in both 1+1D and 3+1D with residuals below 10^{-15}.

4 Einstein's equations

4.1 The derivation

We now have all the ingredients to carry out Jacobson's argument [1] with capacity-derived inputs. The logic is:

1. **Temperature** (from capacity, Section 2.2): $T = \kappa/(2\pi)$.
2. **Entropy** (from capacity, Section 3.2): $S = \eta A$ with $\eta = 1/(4G)$.
3. **Clausius relation** (from entanglement first law, Section 3.4): $\delta Q = T\,dS$.

None of these assume any field equation.

Consider a local Rindler horizon generated by a null vector k^a at an arbitrary spacetime point. The boost Killing vector is $\chi^a = \kappa\,\lambda\,k^a$ near the bifurcation surface, where λ is the affine parameter.

Step 4: Heat flux. The matter energy flux through the horizon is

$$\delta Q = \int_H T_{ab}\,\chi^a\,d\Sigma^b = \frac{\kappa}{2}\,T_{ab}\,k^a k^b\,\delta\lambda^2\,A_\perp\,, \tag{18}$$

where $A_\perp$ is the transverse cross-sectional area.

Step 5: Area change from the Raychaudhuri equation. The Raychaudhuri equation for the expansion θ of the null congruence is

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}\,k^a k^b\,. \tag{19}$$

This is a theorem of differential geometry, not general relativity. Near the bifurcation surface ($\theta = 0$, $\sigma_{ab} = 0$), the area change is

$$\delta A = -R_{ab}\,k^a k^b \cdot \frac{\delta\lambda^2}{2}\,A_\perp\,. \tag{20}$$

Step 6: The null constraint. Combining $\delta Q = T\,dS = T\,\eta\,\delta A$ with Steps 4 and 5, the κ, $\delta\lambda^2$, and $A_\perp$ factors cancel:

$$R_{ab}\,k^a k^b = \frac{2\pi}{\eta}\,T_{ab}\,k^a k^b\,. \tag{21}$$

With $\eta = 1/(4G)$, this becomes

$$R_{ab}\,k^a k^b = 8\pi G\,T_{ab}\,k^a k^b \qquad \text{for all null } k^a\,. \tag{22}$$

Step 7: Tensor reconstruction. If a symmetric tensor Φ_{ab} satisfies $\Phi_{ab}\,k^a k^b = 0$ for all null vectors k^a, then $\Phi_{ab} = f\,g_{ab}$ for some scalar f (an algebraic identity). Applied to $(R_{ab} - 8\pi G\,T_{ab})$:

$$R_{ab} - 8\pi G\,T_{ab} = f\,g_{ab}\,. \tag{23}$$

Taking the trace and using the Bianchi identity $\nabla^a G_{ab} = 0$ and energy conservation $\nabla^a T_{ab} = 0$ gives $f = -\Lambda - R/2 + 4\pi G\,T$, so that

$$\boxed{G_{ab} + \Lambda\,g_{ab} = 8\pi G\,T_{ab}\,.} \tag{24}$$

This is the full Einstein field equation, with the cosmological constant Λ appearing as an undetermined integration constant.

4.2 Numerical verification

We verify the derivation on a suite of backgrounds. A note on the nature of these checks: because the derivation is algebraic (each step is an identity for KMS states on a given background), the machine-precision residuals below confirm that our code correctly implements the chain of identities. The non-trivial physical content—the claim that nature realises KMS equilibrium on local horizons and that the resulting Clausius relation selects GR—is tested in Section 5.

Null constraint. Equation (22) is verified for vacuum, dust, radiation, and general perfect fluid sources. The maximum residual $|R_{ab}k^a k^b - 8\pi G\, T_{ab}k^a k^b|$ is below 3×10^{-16} in all cases. (This is an algebraic identity once $\eta = 1/(4G)$ is assumed.)

Newton's constant. We extract G from the null constraint by varying the matter content. The result: $G_{\text{extracted}} = 1.000000$ with coefficient of variation $< 10^{-16}$, universal across dust, radiation, and generic perfect fluids. This is a consistency check: G is an input (through η), not a prediction.

Area–entropy proportionality. The entropy density S/L scales linearly with κ (coefficient of variation $< 10^{-16}$), with $dS/d\kappa = 1/12$ exactly (for $\sigma = 8$ in 1+1D). This follows from the Stefan–Boltzmann law.

Raychaudhuri focusing. We verify that $R_{ab}\, k^a k^b \geq 0$ (the null energy condition) for all standard matter types, confirming that horizons focus as expected.

5 Non-GR backgrounds: the Clausius residual as a discriminant

The derivation of Section 4 shows that if the Clausius relation (in the GR form $\delta Q = T\, dS$ with $S = A/(4G)$) holds on all local horizons, then the spacetime satisfies Einstein's equations. We now examine the converse: on backgrounds that satisfy modified field equations, the GR-form Clausius relation fails.

At one level, this is expected by construction. An $f(R)$ background satisfies $f(R)$ field equations, not Einstein's, so the GR Clausius residual is nonzero—just as a solution of the heat equation does not satisfy the wave equation. The non-trivial content is threefold:

1. The residual takes the exact form predicted by the difference between the GR and modified field equations—not a generic failure, but a specific, calculable one (Section 5.1).
2. The residual grows linearly with the deformation parameter, so GR is a non-degenerate zero: any infinitesimal departure is detectable (Section 5.5).
3. The capacity framework automatically extracts the Wald temperature on $f(R)$ backgrounds without being told what gravity theory is in play—a genuinely non-trivial result (Section 5.6).

We define the Clausius residual as

$$\delta(k) = R_{ab}\, k^a k^b - 8\pi G\, T_{ab}\, k^a k^b \,. \tag{25}$$

This vanishes for GR backgrounds and measures the departure from the GR field equations for any alternative.

5.1 $f(R)$ gravity

For $f(R) = R + \alpha R^2$ at constant curvature R, the predicted residual is $\delta = -2\alpha R \cdot R_{ab}\, k^a k^b = f''(R) \cdot R \cdot R_{kk}$. Table 5 shows that the measured residual matches this prediction exactly.

Table 5: Clausius residual for $f(R) = R + \alpha R^2$ backgrounds. The residual scales linearly with α and matches the predicted $f''(R) \cdot R \cdot R_{kk}$ to machine precision. GR ($\alpha = 0$) is the unique theory for which the Clausius relation holds.

α	$f''(R)$	Max residual	Clausius
0.000	0.000	7.1×10^{-15}	Holds
0.001	0.002	1.64×10^{-1}	Fails
0.010	0.020	1.64	Fails
0.100	0.200	16.4	Fails
0.500	1.000	82.1	Fails

5.2 Wrong coupling constant

If Newton's constant is perturbed to $G + \Delta G$, the residual scales linearly: residual = $32.7 \times |\Delta G|$ ($R^2 = 1.000$). Even a 0.1% mismatch in G produces a detectable Clausius violation (residual = 3.3×10^{-2}).

5.3 Wrong equation of state

If the metric is constructed for pressure $p = 0.3$ but the actual pressure is $p = 0.31$ ($\Delta p = 0.01$), the residual is 2.5×10^{-1}. A 1% equation-of-state mismatch is detectable.

5.4 Brans–Dicke and higher-derivative theories

We also test Brans–Dicke theory (varying G, perturbation ε) and higher-derivative corrections (βR^2 added to the action). In both cases, the Clausius relation fails for any nonzero deformation:

Table 6: Clausius violations for Brans–Dicke and higher-derivative theories. GR is the unique survivor in all cases.

Theory	Deformation	Residual	Clausius
GR	—	7.1×10^{-15}	Holds
Brans–Dicke	$\varepsilon = 0.001$	3.3×10^{-2}	Fails
Brans–Dicke	$\varepsilon = 0.01$	3.3×10^{-1}	Fails
Higher-deriv.	$\beta = 0.001$	8.2×10^{-2}	Fails
Higher-deriv.	$\beta = 0.01$	8.2×10^{-1}	Fails

5.5 Scaling analysis

Both $f(R)$ and wrong-G residuals scale linearly with the deformation parameter ($R^2 = 1.000$ in both cases) and vanish at the GR point. The Clausius relation acts as a sharp filter: GR is the unique zero of a function that grows linearly in any direction away from it.

5.6 Wald temperature in $f(R)$

An important subtlety: in $f(R)$ gravity, the physical temperature is the Wald temperature $T_{\text{Wald}} = H/(2\pi f'(R))$, not the geometric $T_{\text{GR}} = H/(2\pi)$. The capacity framework automatically selects T_{Wald}, because the Wightman function encodes the full gravitational dynamics:

Table 7: Temperature extraction on $f(R)$ de Sitter backgrounds. The capacity-derived temperature matches the Wald temperature, not the geometric temperature. $H = 1.0$ throughout.

α	T_{GR}	T_{Wald}	$T_{\text{extracted}}$	Match
0.000	0.159	0.159	0.159	Both
0.001	0.159	0.155	0.155	Wald
0.005	0.159	0.142	0.142	Wald
0.010	0.159	0.128	0.128	Wald
0.050	0.159	0.072	0.072	Wald

The capacity framework "knows" about modified gravity without being told. This raises a natural question: since the framework extracts T_{Wald}, should one not check the *modified* Clausius relation $\delta Q = T_{\text{Wald}}\, dS_{\text{Wald}}$ (with Wald entropy $S_{\text{Wald}} = f'(R)\, A/(4G)$), which *does* hold on $f(R)$ backgrounds?

The answer clarifies the logical status of the non-GR tests. The entropy $S = f(Q_t)$ derived in Section 3 is computed from the quantum field content (the Bose–Einstein spectrum of a free scalar), not from the gravitational action. It gives $S \propto A$, not $S_{\text{Wald}} \propto f'(R)\, A$. The capacity-derived Clausius relation is therefore specifically the GR form. The residuals in Tables 5–6 measure the mismatch between this GR-form relation and the background geometry. They do not show that $f(R)$ gravity is "wrong"—it satisfies its own field equations. They show that GR is the unique theory for which the capacity-derived Clausius relation is self-consistent.

6 Beyond strict equilibrium

The slope law theorem assumes exact KMS equilibrium. Physical horizons are only approximately thermal. How robust is the construction?

We test this using an adiabatically evolving de Sitter spacetime with Hubble parameter $H(t) = H_0\, e^{-\epsilon_H H_0 t}$, where ϵ_H controls the rate of change. The detector response is computed by adiabatic averaging of the instantaneous KMS response.

6.1 Universal scaling

Define the adiabaticity parameter $\eta = \epsilon_H H_0 \sigma$, which measures how much H changes during one measurement. We find that the temperature extraction error is a universal function of η:

The scaling is approximately quadratic: deviation $\sim 0.55\, \eta^{2.15}$. For $\eta < 0.15$ (the Hubble parameter changes by less than $\sim 15\%$ during the measurement), the temperature extraction is accurate to better than 1%. This covers most astrophysical settings.

Table 8: Temperature deviation as a function of the adiabaticity parameter $\eta = \epsilon_H H_0 \sigma$. Different combinations of $(H_0, \epsilon_H, \sigma)$ that give the same η produce identical deviations to four significant figures.

η	Deviation	Power-law fit	Quality
0.00	2.1×10^{-5}	—	Excellent
0.04	8.6×10^{-4}	$\sim 0.55\, \eta^{2.15}$	Excellent
0.08	3.5×10^{-3}		Good
0.16	1.4×10^{-2}		Good
0.40	9.4×10^{-2}		Fair
0.80	4.7×10^{-1}		Poor

6.2 Slope law degradation

The power-law exponent α in $Q_t \propto \kappa^\alpha$ degrades with increasing ϵ_H:

Table 9: Slope law degradation with departure from equilibrium.

ϵ_H	α	$\lvert\alpha - 1\rvert$
0.00	1.000	$\sim 10^{-15}$
0.01	1.006	6.3×10^{-3}
0.02	1.026	2.5×10^{-2}
0.05	1.171	1.7×10^{-1}
0.10	1.621	6.2×10^{-1}

The slope law is robust under small departures from equilibrium, degrading smoothly and predictably.

7 Discussion

7.1 What is new and what is not

We are explicit about the division of labour. The slope law theorem (Theorem 3) is an algebraic consequence of the KMS condition: any thermal detector with a Bose–Einstein response satisfies $Q_t \propto \kappa$. The entropy–capacity connection (Section 3) is a rewriting of the Stefan–Boltzmann law in QFI language. The Jacobson derivation (Section 4) is standard [1].

What is new is twofold. First, the timing capacity $Q_t = F_{\text{timing}}^{1/3}$ is, as far as we know, a new observable: a specific quantum-information quantity that linearises the temperature dependence with zero free parameters, holds universally across detector families (spectral indices $n = 0, 1, 2, 3$), and provides a concrete bridge between quantum Fisher information and horizon thermodynamics. Second, the non-GR tests (Section 5) go beyond the original Jacobson argument by establishing the converse: the GR-form Clausius residual is nonzero for every tested alternative, growing linearly with the deformation parameter. GR is not merely consistent with the Clausius relation—it is the unique zero.

7.2 Relation to Jacobson (1995)

Our derivation follows Jacobson's original logic step-for-step [1]. The thermodynamic inputs (temperature, entropy, Clausius relation) are expressed in terms of the timing capacity rather than assumed phenomenologically, but the gravitational content still enters through the Wightman function $G^+(\Delta\tau)$, which encodes the background geometry. The derivation does not produce Einstein's equations from "nothing"—it identifies a consistency condition (vanishing Clausius residual on all local horizons) that the background must satisfy, and shows that this condition is equivalent to the Einstein equations.

7.3 The Wald entropy question

In $f(R)$ gravity, the correct horizon entropy is the Wald entropy $S_{\text{Wald}} = f'(R)\,A/(4G)$, and the modified Clausius relation $\delta Q = T_{\text{Wald}}\,dS_{\text{Wald}}$ holds on $f(R)$ backgrounds. The capacity framework extracts T_{Wald} automatically (Table 7), but the entropy function $S = f(Q_t)$ derived in Section 3 gives $S \propto A$, not S_{Wald}. This is because S is computed from the field content (the scalar Bose–Einstein spectrum), which is insensitive to the gravitational action. The non-GR residuals in Section 5 therefore measure the difference between $S \propto A$ and $S_{\text{Wald}} \propto f'(R)\,A$—they do not show that $f(R)$ gravity is wrong, but that the capacity-derived entropy selects the GR form of the area law.

7.4 Limitations and open problems

The cosmological constant. The derivation produces Λ as an undetermined integration constant. This is a feature shared with Jacobson's original argument. In a companion paper [15], we show that the log correction to entanglement entropy, combined with the Cai–Kim horizon first law, produces a specific value of Λ.

Continuum only. The results of this paper are in the continuum. In a companion paper [16], we show that the same programme can be carried out on causal sets, where the Wightman function, QFI, entropy, and Ricci curvature are all computed from the causal structure alone.

Free fields. The entropy–capacity relation (14) is derived for free scalar fields. The slope law theorem, however, holds for any KMS state, regardless of interactions. The extension to interacting theories is an important open problem.

Detector model dependence. The constant C_Q depends on the detector switching function and coupling type. The exponent $\alpha = 1$, however, is universal. The physical content of the slope law is that horizons are in KMS equilibrium, not the particular value of C_Q.

8 Conclusion

We have shown that Einstein's field equations are the unique metric theory for which the capacity-derived Clausius relation is self-consistent on all local horizons. The argument uses four ingredients:

1. The Wightman two-point function $G^+(\Delta\tau)$, which encodes the background geometry (quantum field theory on curved spacetime);

2. The timing capacity $Q_t = F_{\text{timing}}^{1/3}$, which extracts temperature from G^+ (quantum information);

3. The entanglement first law $\delta\langle K\rangle = \delta S$ (quantum mechanics);

4. The Raychaudhuri equation (differential geometry).

The gravitational content enters through G^+; the field equations emerge as the condition that the Clausius relation hold simultaneously on all local horizons. Newton's constant is determined by the entropy–area proportionality, and Λ appears as an integration constant.

The timing capacity provides a concrete, calculable bridge between quantum Fisher information and horizon thermodynamics, and the non-GR tests establish that GR is not merely consistent with this bridge but uniquely selected by it.

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References

[1] T. Jacobson, "Thermodynamics of spacetime: the Einstein equation of state," *Phys. Rev. Lett.* **75**, 1260 (1995); arXiv:gr-qc/9504004.

[2] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, 1982).

[3] L. C. B. Crispino, A. Higuchi, and G. E. A. Matsas, "The Unruh effect and its applications," *Rev. Mod. Phys.* **80**, 787 (2008); arXiv:0710.5373.

[4] D. D. Blanco, H. Casini, L.-Y. Hung, and R. C. Myers, "Relative entropy and holography," *JHEP* **2013**, 060 (2013); arXiv:1305.3182.

[5] T. Faulkner, M. Guica, T. Hartman, R. C. Myers, and M. Van Raamsdonk, "Gravitation from entanglement in holographic CFTs," *JHEP* **2014**, 051 (2014); arXiv:1312.7856.

[6] M. Srednicki, "Entropy and area," *Phys. Rev. Lett.* **71**, 666 (1993); arXiv:hep-th/9303048.

[7] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, "Quantum source of entropy for black holes," *Phys. Rev. D* **34**, 373 (1986).

[8] S. N. Solodukhin, "Entanglement entropy of black holes," *Living Rev. Relativ.* **14**, 8 (2011); arXiv:1104.3712.

[9] H. Casini and M. Huerta, "Entanglement entropy in free quantum field theory," *J. Phys. A* **42**, 504007 (2009); arXiv:0905.2562.

[10] E. Bianchi and R. C. Myers, "On the architecture of spacetime geometry," *Class. Quant. Grav.* **31**, 214002 (2014); arXiv:1212.5183.

[11] Planck Collaboration, "Planck 2018 results. VI. Cosmological parameters," *Astron. Astrophys.* **641**, A6 (2020); arXiv:1807.06209.

[12] S. Weinberg, "The cosmological constant problem," *Rev. Mod. Phys.* **61**, 1 (1989).

[13] R.-G. Cai and S. P. Kim, "First law of thermodynamics and Friedmann equations of Friedmann–Robertson–Walker universe," *JHEP* **2005**, 050 (2005); arXiv:hep-th/0501055.

[14] W. G. Unruh, "Notes on black-hole evaporation," *Phys. Rev. D* **14**, 870 (1976).

[15] "Cosmological constant from entanglement entropy: a derivation via Jacobson–Cai-Kim horizon thermodynamics" (2025).

[16] "Deriving Einstein's equations on causal sets: a numerical demonstration" (2025).