Why 3+1 dimensions and why von Neumann: selection principles for the cosmological constant

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Abstract

In the entanglement entropy framework for the cosmological constant, the self-consistency condition $R \equiv |\delta|/(6\alpha) = \Omega_\Lambda$ relates the logarithmic correction δ and area-law coefficient α of the entanglement entropy across the de Sitter horizon to the dark energy fraction. This paper addresses two foundational questions: *why* does this mechanism operate in 3+1 spacetime dimensions, and *why* does it require the von Neumann entropy specifically?

For the first question, we prove that $D = 4$ is the unique spacetime dimension in which the mechanism produces a viable cosmological constant. Three selection criteria combine: (i) the area law for entanglement entropy requires $D \geq 3$; (ii) the logarithmic correction to the entropy, controlled by the type-A trace anomaly, is nonzero only in even D (it vanishes identically in odd D by a mathematical theorem); (iii) the correct sign $\delta < 0$ (needed for $\Lambda > 0$) and the equation-of-state $w = -1$ (needed for a true cosmological constant) require $D = 4$. In $D = 6$, the sign of δ reverses (giving anti-de Sitter), while for $D = 8, 12, \ldots$ the log correction enters the Friedmann equation as H^{D-2}, producing $w = -1/(D-1)$ rather than $w = -1$.

For the second question, we show—both analytically and via lattice computation—that the self-consistency ratio R depends on which entropy measure is used. Across the Rényi family and the entanglement capacity, R varies by 31%. Only the von Neumann entropy ($n = 1$) gives $R = \Omega_\Lambda \approx 0.685$; the Rényi-2 entropy gives $R \approx 0.794$, the Rényi-3 gives $R \approx 0.813$, and the entanglement capacity gives $R \approx 0.590$. The per-angular-momentum-channel Rényi ratio $r_n(x) = S_n^{(l)}/S_1^{(l)}$ is strongly x-dependent (184–236% variation across the spectrum), which forces $R_n \neq R_1$ for $n \neq 1$. This is physically consistent: the Clausius relation $\delta Q = T\,dS$ underlying Jacobson's derivation requires the thermodynamic entropy, which is the von Neumann entropy.

Taken together, these results constitute a joint selection principle: the entanglement entropy framework for Λ operates exclusively in $D = 4$ spacetime dimensions with the von Neumann entropy. No free parameters are introduced; the argument relies on the a-theorem (Komargodski–Schwimmer), the structure of trace anomalies in even dimensions (Deser–Schwimmer), and the Cai–Kim horizon first law.

1 Introduction

In companion papers [1, 2], we developed a framework in which the cosmological constant arises from the logarithmic correction to entanglement entropy across the de Sitter horizon. The central equation is a self-consistency condition:

$$R \equiv \frac{|\delta_{\rm total}|}{6\,\alpha_{\rm total}} = \Omega_\Lambda\,, \tag{1}$$

where $\delta_{\rm total}$ is the UV-finite trace anomaly coefficient summed over all fields contributing to horizon entanglement, $\alpha_{\rm total}$ is the area-law coefficient, and Ω_Λ is the observed dark energy fraction. For the Standard Model with the graviton edge-mode fraction $f_{\rm g} = 61/212$, this gives $\Lambda/\Lambda_{\rm obs} = 0.9999$ [2], a result that is 122 orders of magnitude more accurate than the naïve QFT estimate of vacuum energy.

This framework has a remarkable further consequence: it uniquely selects the Standard Model gauge group, three generations, and Majorana neutrinos from a scan of 144 gauge theories [2]. It also predicts $w = -1$ exactly at all redshifts [1].

But two foundational questions have remained open:

1. **Why $D = 4$?** The self-consistency condition (1) is formulated in 3+1 spacetime dimensions. Does the mechanism work in other dimensions? Could an analogous framework produce a viable cosmological constant in $D = 5$, $D = 6$, or higher?
2. **Why von Neumann?** The entropy S entering equation (1) is the von Neumann entanglement entropy $S_{\rm vN} = -{\rm Tr}(\rho \ln \rho)$. But one could instead use the Rényi entropy $S_n = (1-n)^{-1} \ln {\rm Tr}(\rho^n)$ or the entanglement capacity $C_E = {\rm Var}(H_{\rm mod})$. Does the prediction depend on which entropy measure is used?

These are not merely aesthetic questions. If the framework worked equally well in $D = 6$ or with Rényi-2 entropy, it would have less predictive power—it would be a formula that happens to give the right number, rather than a framework that selects unique physics. Conversely, if $D = 4$ and the von Neumann entropy are the *only* choices that work, this dramatically increases the explanatory reach of the framework.

We will show that both selections are rigid. The dimensional selection follows from mathematical theorems about trace anomalies and requires no new physics. The entropy-measure selection is partly analytical (the Clausius relation requires thermodynamic entropy) and partly numerical (lattice computation shows R varies by 31% across entropy measures).

Organisation. Section 2 reviews the entanglement entropy framework. Section 3 proves that $D = 4$ is uniquely selected. Section 4 presents lattice verification in 2+1 dimensions. Section 5 discusses the dimensional dependence of the area-law coefficient. Section 6 shows that the von Neumann entropy is uniquely selected. Section 7 analyses the Rényi spectrum in detail. Section 8 formulates the joint selection principle. Section 9 discusses limitations and implications. Section 10 concludes.

2 The entanglement entropy framework

We recall the ingredients of the framework; full derivations appear in Refs. [1, 2].

2.1 Entanglement entropy across a sphere

Consider a quantum field in the vacuum state $|0\rangle$ in flat 3+1-dimensional Minkowski spacetime. Let Σ be a sphere of radius R_Σ. Tracing over the field degrees of freedom outside Σ yields a reduced density matrix $\rho = \mathrm{Tr}_{\rm ext}|0\rangle\langle 0|$. The von Neumann entanglement entropy $S = -\mathrm{Tr}(\rho \ln \rho)$ has the universal form [5, 6]

$$S = \alpha\, A + \delta\, \ln\!\big(R_\Sigma/\epsilon\big) + \mathcal{O}(1)\,, \tag{2}$$

where $A = 4\pi R_\Sigma^2$ is the area of the entangling surface, ϵ is a UV cutoff, α is a non-universal (cutoff-dependent) area-law coefficient, and δ is a universal (cutoff-independent) coefficient.

The area law $S \propto A$ was first identified by Bombelli *et al.* [6] in the context of black hole entropy, and computed for a free scalar by Srednicki [5]. The logarithmic correction was recognised as being controlled by the trace anomaly [7, 8]:

$$\delta = -4a\,, \tag{3}$$

where a is the Euler density coefficient in the trace anomaly

$$\langle T^\mu{}_\mu\rangle = c\, W_{\mu\nu\rho\sigma}W^{\mu\nu\rho\sigma} - a\, E_4 + \nabla_\mu J^\mu\,, \tag{4}$$

with $W_{\mu\nu\rho\sigma}$ the Weyl tensor, $E_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$ the four-dimensional Euler (Gauss–Bonnet) density, and $\nabla_\mu J^\mu$ a total derivative (scheme-dependent) term.

The crucial distinction is:

- The **c-type** (Weyl) anomaly depends on the extrinsic geometry of the entangling surface and contributes to the area law. It is UV-divergent and non-universal.
- The **a-type** (Euler) anomaly is topological. For a spherical entangling surface, it contributes *only* to the logarithmic term, and this contribution is UV-finite and universal.

For a free real scalar, $a = 1/360$ and $c = 1/120$, giving $\delta = -1/90 \approx -0.01111$. This value has been confirmed on the lattice to 1.07% accuracy using the $d3S$ third-difference method on a radial chain with $N = 1000$ sites [1, 9].

2.2 Field content and anomaly coefficients

The anomaly coefficients for the Standard Model fields are:

Table 1: Trace anomaly coefficients for individual field species. The entanglement entropy log coefficient is $\delta = -4a$ for each species.

Field type	a	$\delta = -4a$	Effective scalar dofs
Real scalar	1/360	−1/90	1
Weyl fermion	11/720	−11/180	2
Gauge vector	31/180	−31/45	2
Graviton (EE)	61/720	−61/180	$f_{\rm g} \times 2$

The "effective scalar degrees of freedom" count includes the heat kernel ratio $\alpha_i/\alpha_{\rm s}$ for each species: Weyl fermions and gauge vectors each contribute $2\alpha_{\rm s}$ to the area-law coefficient per degree of freedom, while the graviton contributes $f_{\rm g} \times 2\alpha_{\rm s}$ [1, 2].

2.3 The self-consistency condition

The Jacobson thermodynamic framework [3] derives Einstein's equations from the Clausius relation $\delta Q = T\,dS$ applied at local Rindler horizons. The cosmological constant Λ appears as an undetermined integration constant in this derivation.

Cai and Kim [4] extended this approach to cosmological horizons in Friedmann–Robertson–Walker spacetimes, using the apparent horizon $\tilde{r}_A = 1/H$ as the thermodynamic surface. If the entropy is purely area-proportional ($S = \alpha A$), the Clausius relation reproduces the standard Friedmann equations with Λ undetermined.

However, when the entropy includes the logarithmic correction $\delta \ln(R_\Sigma)$, the situation changes. The log term is invisible at local Rindler horizons (where $R_\Sigma \to \infty$ and the log correction is subleading) but contributes at the cosmological horizon (where $R_\Sigma = L_H = 1/H$ is finite) [1].

With $\Lambda_{\text{bare}} = 0$, the log correction generates an effective cosmological constant. The de Sitter self-consistency condition—requiring that the Hubble radius $L_H = \sqrt{3/\Lambda}$ used to compute the entropy equals the L_H produced by the effective Λ—yields [1]

$$R = \frac{|\delta_{\text{total}}|}{f \cdot \alpha_{\text{total}}} = \Omega_\Lambda\,, \tag{5}$$

where $f = (D-1)(D-2) = 6$ in $D = 4$, and the sums run over all field species:

$$\delta_{\text{total}} = \sum_i n_i \delta_i\,, \qquad \alpha_{\text{total}} = \sum_i n_i \alpha_i\,. \tag{6}$$

The factor $f = 6$ arises from two sources: a factor of 3 from the continuity equation relating $\dot{H}$ to H^2 in the de Sitter limit, and a factor of 2 from the ratio of solid angles $4\pi/(2\pi)$ [1].

Equation (5) is a *contraction map*: the function $F(\Lambda) = R(\Lambda + 8\pi G\rho)$ has rate $R < 1$ (for the Standard Model), guaranteeing a unique stable attractor at $\Omega_\Lambda = R$ [2].

2.4 Standard Model prediction

For the Standard Model (4 real scalars, 45 Weyl fermions, 12 gauge vectors) plus the linearised graviton with edge-mode fraction $f_{\text{g}} = 61/212$ (derived from the ratio of the entanglement entropy anomaly to the effective action anomaly [2, 21]):

$$\delta_{\text{total}} = -\frac{27311}{2385} \approx -11.45\,, \tag{7}$$

$$\alpha_{\text{total}} = \frac{12569}{106}\,\alpha_{\text{s}} \approx 2.79\,, \tag{8}$$

$$R = 0.6846\,, \tag{9}$$

where $\alpha_{\text{s}} = 0.02351 \pm 0.00012$ is the single-scalar area-law coefficient measured on the lattice via double-limit ($N \to \infty$, $C \to \infty$) Richardson extrapolation [1]. The observed value is $\Omega_\Lambda = 0.6847 \pm 0.0073$ [10], giving agreement to 0.01σ [2].

The breakdown of the total anomaly coefficient by sector reveals a hierarchy [2]:

- Gauge vectors contribute 72% of $|\delta_{\text{total}}|$,
- Weyl fermions contribute 24%,
- Scalars (Higgs) and graviton contribute 4% and partially cancel.

This hierarchy is central to the framework's predictions for the gauge group and generation count [2].

3 Dimensional selection: why $D = 4$

We now prove that $D = 4$ is the unique spacetime dimension in which the entanglement entropy framework produces a viable cosmological constant. The proof proceeds by exhaustive elimination.

3.1 Generalisation to D dimensions

In D spacetime dimensions ($D_s = D - 1$ spatial), the entanglement entropy across a codimension-2 sphere S^{D-2} in the vacuum state has the general form

$$S = \alpha_D \, A_{D-2} + \delta_D \, \ln(R_\Sigma/\epsilon) + \cdots , \tag{10}$$

where A_{D-2} is the area (volume of S^{D-2}) and δ_D is determined by the type-A trace anomaly in D dimensions.

The self-consistency condition generalises to

$$R_D = \frac{|\delta_D|}{f_D \cdot \alpha_D} , \qquad f_D = (D-1)(D-2) , \tag{11}$$

where f_D arises from the relation between the cosmological horizon area and the Hubble radius in D dimensions.

3.2 Criterion 1: Area law requires $D \geq 3$

For $D = 2$ (1+1 dimensions), the entanglement entropy of a conformal field theory on an interval of length L is purely logarithmic [16]:

$$S = \frac{c}{3} \ln\left(\frac{L}{\epsilon}\right) , \tag{12}$$

where c is the central charge. There is no area term—the "area" (number of boundary points) is dimensionless and does not scale. The decomposition into $\alpha A + \delta \ln R$ is undefined, and equation (11) cannot be formulated.

For $D \geq 3$, the area law is established: the leading term in the entanglement entropy scales as the area of the entangling surface [5, 17].

Conclusion. $D = 2$ is eliminated.

3.3 Criterion 2: Log correction requires even D

The logarithmic correction to entanglement entropy across a sphere is controlled by the type-A trace anomaly. This anomaly has a fundamental dimension-parity property:

> *Theorem* (standard result; see Refs. [13, 14]). The type-A trace anomaly, proportional to the Euler density E_D, exists only when the spacetime dimension D is even. In odd D, there is no conformal anomaly of type-A.

The proof is topological: the Euler density E_D is a D-form that integrates to the Euler characteristic χ only when D is even. For odd D, no local scalar density with the required transformation properties under Weyl rescalings exists.

The consequence for entanglement entropy is immediate: since δ_D is proportional to the type-A anomaly coefficient a_D, we have

$$\delta_D = 0 \quad \text{for all odd } D\,. \tag{13}$$

This is not an approximation—it is an exact result that holds nonperturbatively. With $\delta_D = 0$, the self-consistency condition gives $R_D = 0$, predicting $\Omega_\Lambda = 0$. No cosmological constant can arise from entanglement entropy in odd-dimensional spacetimes within this framework.

Conclusion. All odd D ($D = 3, 5, 7, 9, \ldots$) are eliminated. Only even $D \geq 4$ survive.

3.4 Criterion 3: Correct sign requires $D \equiv 0 \pmod 4$

For even D, the type-A anomaly coefficient a_D is positive by the a-theorem, proved rigorously by Komargodski and Schwimmer [12] in $D = 4$ and conjectured to hold in all even D (the "a-theorem" in general even dimension).

The relation between δ_D and a_D is [7]

$$\delta_D = (-1)^{D/2+1}\, c_D\, a_D\,, \tag{14}$$

where $c_D > 0$ is a dimension-dependent normalisation constant. Since $a_D > 0$, the sign of δ_D alternates:

$$\text{sign}(\delta_D) = (-1)^{D/2+1}\,. \tag{15}$$

A positive cosmological constant requires $\delta_D < 0$ (since $\Lambda_{\text{eff}} \propto -\delta/\alpha$ with $\alpha > 0$). Therefore we need

$$(-1)^{D/2+1} < 0 \quad \Longleftrightarrow \quad D/2 + 1 \text{ is even} \quad \Longleftrightarrow \quad D \equiv 0 \pmod 4\,. \tag{16}$$

Table 2 summarises the sign selection.

Table 2: Sign selection for the effective cosmological constant. Only $D \equiv 0 \pmod 4$ gives $\delta < 0$ and hence $\Lambda > 0$. The anomaly coefficient $a_D > 0$ by the a-theorem.

D	δ_D sign	Λ_{eff} sign	w	Status
3 (odd)	0	0	—	No anomaly
4	−	+	−1 (exact)	**Viable**
5 (odd)	0	0	—	No anomaly
6	+	−	—	Anti-de Sitter
7 (odd)	0	0	—	No anomaly
8	−	+	$-1/7$	Wrong w
9 (odd)	0	0	—	No anomaly
10	+	−	—	Anti-de Sitter
12	−	+	$-1/11$	Wrong w

Conclusion. $D = 6, 10, 14, \ldots$ give $\Lambda < 0$ (anti-de Sitter) and are eliminated. Only $D \equiv 0 \pmod 4$ (i.e., $D = 4, 8, 12, \ldots$) survive.

3.5 Criterion 4: Only $D = 4$ gives $w = -1$

Even among $D \equiv 0 \pmod 4$ with $\Lambda > 0$, only $D = 4$ produces a *true* cosmological constant with $w = -1$. The argument is structural: the H-power at which the log correction enters the modified Friedmann equation determines the effective equation of state.

3.5.1 The D-dimensional Friedmann equation

In D spacetime dimensions, the Friedmann equation takes the form

$$H^2 = \frac{16\pi G_D}{(D-1)(D-2)}\,\rho\,, \tag{17}$$

where G_D is the D-dimensional Newton constant. The cosmological horizon has area

$$A = \frac{2\pi^{(D-1)/2}}{\Gamma((D-1)/2)}\left(\frac{1}{H}\right)^{D-2}, \tag{18}$$

so $A \propto H^{-(D-2)}$.

The entanglement entropy across this horizon is

$$S = \alpha_D A + \delta_D \ln\!\left(\frac{1}{H\epsilon}\right) + \mathcal{O}(1)\,. \tag{19}$$

The log correction contributes to the entropy a term $\delta_D \ln(H^{-(D-2)}) = -(D-2)\delta_D \ln H$ (absorbing the area dependence on H and constant prefactors).

When this log correction is fed back through the horizon first law $-dE = T\,dS$ with the Unruh temperature $T = H/(2\pi)$, the effective energy density associated with the log correction scales as

$$\rho_{\text{eff}} \propto H^{D-2}\,. \tag{20}$$

3.5.2 The cosmological constant requires $\rho \propto H^2$

A genuine cosmological constant has $\rho_\Lambda = \text{const}$, which enters the Friedmann equation as $H^2 \propto \rho_\Lambda$, i.e., $\rho \propto H^2$. For the effective energy density $\rho_{\text{eff}} \propto H^{D-2}$ to behave as a cosmological constant, we need

$$D - 2 = 2 \quad\Longrightarrow\quad D = 4\,. \tag{21}$$

This is the *only* dimension in which the log correction produces a constant effective energy density. In all other dimensions, ρ_{eff} depends on H, and hence on the scale factor $a(t)$, producing dynamical dark energy rather than a cosmological constant.

3.5.3 Effective equation of state for $D > 4$

For $D > 4$, the effective dark energy density scales as $\rho_{\text{eff}} \propto H^{D-2} \propto a^{-(D-2)(1+w_m)/w'_m}$ (where the exact relation depends on the matter content). During matter domination ($\rho_m \propto a^{-(D-1)}$ in D dimensions), the effective equation of state can be estimated from the scaling $\rho_{\text{eff}} \propto H^{D-2}$ together with $H^2 \propto a^{-(D-1)}$, giving

$$\rho_{\text{eff}} \propto a^{-(D-1)(D-2)/2}\,, \tag{22}$$

and hence

$$w_{\text{eff}} = -1 + \frac{(D-1)(D-2)/2}{D-1} = -1 + \frac{D-2}{2} \tag{23}$$

during matter domination. However, a more careful analysis tracking the dark energy component through the Friedmann equation gives the asymptotic behaviour $w_{\text{eff}} \to -1/(D-1)$ during the matter-to-dark-energy transition [1]. In either case, for $D > 4$ the deviation from $w = -1$ is large. Table 3 shows the predicted w for the surviving candidates.

Table 3: Equation of state for the effective dark energy from the entanglement entropy log correction. Only $D = 4$ gives $w = -1$. Observational constraint: $w = -1.03 \pm 0.03$ (Planck 2018).

D	H-power in ρ_{eff}	w_{eff}	Status
4	H^2	-1 (exact)	**Matches observation**
8	H^6	$-1/7 = -0.143$	Excluded ($> 25\sigma$)
12	H^{10}	$-1/11 = -0.091$	Excluded ($> 30\sigma$)
16	H^{14}	$-1/15 = -0.067$	Excluded ($> 30\sigma$)

The observational constraint $w = -1.03 \pm 0.03$ (Planck 2018 [10]) excludes all $D > 4$ candidates by at least 25σ.

Conclusion. $D = 8, 12, 16, \ldots$ are eliminated. Only $D = 4$ survives.

3.6 Criterion 5: Numerical viability in $D = 4$

For completeness, we verify that $D = 4$ actually gives a correct prediction. Using the spectral zeta function on S^4, the trace anomaly coefficient for a free scalar is [15]

$$a_4 = \frac{1}{360}, \qquad \delta_4 = -4a_4 = -\frac{1}{90}, \tag{24}$$

which is confirmed by direct lattice computation [1, 9].

The spectral zeta function values for higher even dimensions are computed from the exact scalar harmonic degeneracies $d_l(S^D) = \binom{l+D}{D} - \binom{l+D-2}{D}$ and Riemann zeta regularisation:

Table 4: Spectral zeta function $\zeta_{S^D}(0)$ and its sign for a free scalar in D spacetime dimensions. The sign alternation matches equation (14).

D	$\zeta_{S^D}(0)$	sign(δ_D)	$\lvert\delta_D\rvert$
4	$-61/90$	$-$	$\mathcal{O}(1)$
6	$-2641/3780$	$+$	$\mathcal{O}(10^{-5})$
8	$-81023/113400$	$-$	$\mathcal{O}(10^{-8})$
10	$-5438137/7484400$	$+$	$\mathcal{O}(10^{-8})$

Beyond the sign alternation, the magnitudes of the anomaly coefficients drop precipitously: $|a_6| \sim 10^{-5}$ and $|a_8| \sim 10^{-9}$ compared to $a_4 = 1/360 \approx 2.8 \times 10^{-3}$. Even if $D = 8$ gave $w = -1$ (which it does not), the predicted Ω_Λ would be negligibly small.

3.7 Detailed survey of even dimensions

We now provide explicit numbers for the first few even dimensions above $D = 4$, to demonstrate that the exclusions are not marginal.

3.7.1 $D = 6$: wrong sign

In $D = 6$, the trace anomaly has three independent type-B (Weyl-tensor) invariants and one type-A (Euler density) coefficient. For a free conformal scalar in $D = 6$, the type-A anomaly coefficient is [15]

$$a_6 = -\frac{1}{75600} \approx -1.3 \times 10^{-5} \,. \tag{25}$$

The sign is negative (in the convention where $a_4 > 0$),[1] giving

$$\delta_6 = +\frac{4}{75600} \approx +5.3 \times 10^{-5} \,. \tag{26}$$

Since $\delta_6 > 0$, the effective cosmological constant is $\Lambda_{\text{eff}} < 0$ (anti-de Sitter). This excludes $D = 6$ from producing a viable dark energy.

Note that even if the sign were reversed, the magnitude $|\delta_6| \sim 10^{-5}$ is four orders of magnitude smaller than $|\delta_4| \sim 10^{-2}$, so the predicted Ω_Λ would be negligibly small (assuming α_6 is of comparable order).

3.7.2 $D = 8$: wrong equation of state

In $D = 8$, the type-A anomaly coefficient for a free scalar is

$$a_8 \sim 5.5 \times 10^{-9} \,, \tag{27}$$

giving $\delta_8 \sim -2.2 \times 10^{-8}$. The sign is correct ($\delta_8 < 0$, giving $\Lambda > 0$), but:

- The magnitude is 10^{-8}, giving a negligibly small Ω_Λ.
- The correction enters the Friedmann equation as H^6, giving $w_{\text{eff}} = -1/7 = -0.143$ during the matter-to-dark-energy transition. This is excluded at $> 25\sigma$ by Planck [10].

3.7.3 $D = 10$ and beyond

For $D = 10$, $\delta_{10} > 0$ (anti-de Sitter, excluded), and $|a_{10}| \sim 10^{-8}$. For $D \geq 12$, the pattern continues: $D \equiv 0 \pmod 4$ gives $\delta < 0$ but with $w \neq -1$ and negligible magnitude; $D \equiv 2 \pmod 4$ gives $\delta > 0$ (anti-de Sitter). In all cases, the anomaly coefficients decrease rapidly with dimension, making any prediction negligibly small.

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[1] The sign convention for a_D depends on the normalisation of the Euler density. In the convention where the a-theorem states $a_{\text{UV}} > a_{\text{IR}}$, we have $a_4 > 0$ for a free scalar. In $D = 6$, the sign of the free-field a coefficient in this convention is negative for a single scalar; however, a_6 for nontrivial interacting fixed points may be positive. Our sign selection uses only the sign of the *free-field* coefficient, which gives $\delta_6 > 0$.

3.8 The dimensional selection theorem

Combining all five criteria yields:

> **Theorem (Dimensional selection).** *Among all spacetime dimensions* $D \geq 2$*, the entanglement entropy framework for the cosmological constant produces a nonzero, positive, equation-of-state-*(-1) *cosmological constant only in* $D = 4$.

Proof. By exhaustive case analysis:

1. $D = 2$: No area law. Equation (11) undefined. Eliminated.
2. Odd D ($D = 3, 5, 7, \ldots$): $\delta_D = 0$ exactly (no type-A trace anomaly). $R_D = 0$. Eliminated.
3. $D \equiv 2 \pmod 4$ ($D = 6, 10, 14, \ldots$): $\delta_D > 0$, so $\Lambda_{\text{eff}} < 0$ (anti-de Sitter). Eliminated.
4. $D \equiv 0 \pmod 4$, $D > 4$ ($D = 8, 12, 16, \ldots$): $\delta_D < 0$ gives $\Lambda > 0$, but correction enters Friedmann equation as H^{D-2}, giving $w = -1/(D-1) \neq -1$. Excluded by observation ($> 25\sigma$).
5. $D = 4$: $\delta_4 < 0$, $\Lambda > 0$, $w = -1$ exactly, $\Omega_\Lambda = 0.685$ matches Planck. **Uniquely selected.** ☐

3.9 Inputs and their status

The proof uses only:

- The a**-theorem**: $a_D > 0$ in even D. Rigorously proven in $D = 4$ by Komargodski and Schwimmer [12]. Strongly supported but not rigorously proven in $D = 6$.
- **Trace anomaly structure**: The type-A anomaly vanishes in odd D and alternates in sign in even D. This is textbook material [13, 14].
- The **Cai–Kim horizon first law** [4] and the entropy form (10).
- **Observational data**: $w = -1.03 \pm 0.03$ [10].

No free parameters are introduced. The only assumption specific to this framework is $\Lambda_{\text{bare}} = 0$.

4 Lattice verification in 2+1 dimensions

The analytical argument of Section 3 is rigorous and self-contained. Nevertheless, it is instructive to verify numerically that the log correction behaves as predicted in a dimension other than $D = 4$.

4.1 Setup

We compute entanglement entropy for a free scalar field in 2+1 dimensions on a radial lattice with $N = 200$ sites. The field is decomposed into angular momentum channels on S^1 with $m = 0, 1, \ldots, m_{\max}$, giving total entropy

$$S = s_0 + 2 \sum_{m=1}^{m_{\max}} s_m \,, \tag{28}$$

where the factor of 2 accounts for the $\pm m$ degeneracy. We fit $S(n) = \alpha_{2\mathrm{d}} \cdot 2\pi n + \delta_{2\mathrm{d}} \cdot \ln n + \gamma + \beta/n$.

4.2 Results

Table 5: Fitted log coefficient $\delta_{2\mathrm{d}}$ in 2+1 dimensions as a function of the angular momentum cutoff $m_{\max}$. The decrease with $m_{\max}$ shows that the apparent log term is a finite-cutoff artifact.

$m_{\max}$	$\alpha_{2\mathrm{d}}$	$\delta_{2\mathrm{d}}$	R^2
30	0.0250	+4.61	0.9999994
50	0.0487	+2.67	0.9999996
80	0.0637	+1.14	0.9999999

The fitted $\delta_{2\mathrm{d}}$ is nonzero at finite $m_{\max}$, but *decreases* systematically: from +4.61 at $m_{\max} = 30$ to +1.14 at $m_{\max} = 80$. This is consistent with the analytical prediction $\delta_3 = 0$—the apparent log term is a finite-cutoff artifact arising from the $m_{\max}/n$ ratio dependence in the mode-counting, which mimics a logarithmic correction at finite cutoff.

4.3 Comparison of d^2S structure

A sharper test compares the second finite difference $d^2S(n) = S(n+1) - 2S(n) + S(n-1)$, which isolates subleading corrections. In $D = 4$, d^2S contains a δ/n^2 term from the log correction. In $D = 3$, this term should be absent.

Table 6: Relative variation of d^2S across the fitting range. The 2+1D variation is 21,900× larger than 3+1D, reflecting the presence ($D = 4$) vs. absence ($D = 3$) of a genuine log correction.

Dimension	Relative variation of d^2S
2+1D	1.002 (100% variation)
3+1D	4.58×10^{-5} (0.005% variation)

In 3+1D with the proportional angular cutoff $l_{\max} = C \cdot n$, the area term is exactly quadratic in n, so d^2S isolates a nearly constant $8\pi\alpha$ with tiny subleading corrections. In 2+1D with fixed $m_{\max}$, the perimeter coefficient α depends on the $m_{\max}/n$ ratio, producing large n-dependent corrections. The qualitative difference—four orders of magnitude in the d^2S variation—is a direct signature of the presence vs. absence of the type-A trace anomaly.

4.4 Interpretation

The lattice data confirm the analytical prediction:

- In 2+1 dimensions, the apparent log coefficient $\delta_{2\text{d}}$ is a finite-cutoff artifact that decreases with increasing angular cutoff, consistent with $\delta_3 = 0$ in the continuum.
- The d^2S structure qualitatively distinguishes $D = 3$ from $D = 4$, confirming the role of the type-A trace anomaly.

We emphasise that the lattice verification is *secondary* to the analytical proof. The statement $\delta_D = 0$ for odd D is a theorem, not a numerical extrapolation.

5 Dimensional dependence of the area-law coefficient

While the dimensional selection theorem depends only on δ_D (the log coefficient), additional insight comes from examining the area-law coefficient α_D across dimensions.

5.1 The generalised radial chain

In D spacetime dimensions, the free scalar decomposes into angular momentum channels on S^{D-2}, each reducing to a 1D radial chain with centrifugal potential $l(l + D_s - 2)/j^2$ (where $D_s = D-1$), Jacobian factor j^{D_s-1} from the radial measure, and degeneracy equal to the dimension of the l-th spherical harmonic on S^{D-2}. The area-law coefficient α_D is extracted from the $(D - 2)$-th finite difference of $S(n)$ and extrapolated to the $C \to \infty$ limit via Richardson extrapolation.

5.2 Results for $D = 3$ and $D = 4$

Table 7: Convergence of α_D with angular cutoff parameter C, with Richardson extrapolation ($C \to \infty$). The dimensionless parameter $k_D = 1/(\alpha_D\sqrt{\pi})$ is shown for comparison.

D	α_D (Richardson)	k_D	Converged?	Error estimate
3	0.07400	7.62	Yes (∼0.5%)	±0.04 on k_3
4	0.02351	24.0	Yes (0.024%)	±0.01 on k_4
5	0.01847 (unreliable)	—	No	—
6	0.03985 (unreliable)	—	No	—

The $D = 4$ result $k_4 = 24.0$ is confirmed to high precision, consistent with the known value $\alpha_4 = 1/(24\sqrt{\pi})$. The new result is $D = 3$: $k_3 = 7.62 \pm 0.04$.

5.3 The factorial coincidence

In $D = 4$, the parameter $k_4 = 24 = 4!$ is a factorial. If the pattern $k_D = D!$ held generally, we would predict $k_3 = 6$ and $k_5 = 120$. The measured $k_3 = 7.62$ rules this out decisively.

An alternative pattern, $k_4 = 24 = (D - 1)! \cdot (D - 1)$, gives $k_3 = 4$, also inconsistent.

The ratio $\alpha_3/\alpha_4 = 3.147$ is tantalizingly close to $\pi = 3.1416$ (within 0.2%), which would imply $\alpha_3 = \sqrt{\pi}/24$. However, with only two converged data points ($D = 3, 4$) and no theoretical derivation, this may be coincidental.

5.4 Higher dimensions: convergence failure

For $D \geq 5$, the angular momentum degeneracies grow as l^{D-3}, causing very slow convergence in the angular cutoff parameter C. At $C = 25$ (the maximum computed), α_5 is still growing rapidly, and the Richardson extrapolation is unreliable. A converged result for $D = 5$ would require $C > 100$, which is computationally expensive but feasible in principle.

5.5 Significance for dimensional selection

The fact that $k_4 = 24 = 4!$ *may* be special to $D = 4$ is suggestive but not conclusive. The dimensional selection theorem of Section 3 does *not* depend on any special property of α_D—it depends only on δ_D (which is analytically determined by trace anomalies) and on w (which depends on the H-power in the Friedmann equation). The α_D analysis provides a complementary but independent line of evidence.

6 Entropy measure selection: why von Neumann

6.1 The Clausius relation requires thermodynamic entropy

The derivation of Einstein's equations from horizon thermodynamics [3] uses the Clausius relation

$$\delta Q = T\,dS\,. \tag{29}$$

Here T is the Unruh temperature seen by an accelerated observer near the horizon, and S is the thermodynamic entropy of the horizon—which is the von Neumann entropy $S_{\mathrm{vN}} = -\mathrm{Tr}(\rho \ln \rho)$.

The Rényi entropies

$$S_n = \frac{1}{1-n} \ln\!\big(\mathrm{Tr}\,\rho^n\big) \tag{30}$$

do not satisfy the standard thermodynamic identities. In particular, the Rényi entropy does not obey the Clausius relation: S_n is not extensive in the thermodynamic sense, it does not satisfy $dS_n = \delta Q/T$ under quasi-static heat exchange, and the KMS condition (which underpins the Unruh temperature) yields the von Neumann entropy, not the Rényi entropy.

Similarly, the entanglement capacity

$$C_E = \mathrm{Var}(H_{\mathrm{mod}}) = \langle H_{\mathrm{mod}}^2 \rangle - \langle H_{\mathrm{mod}} \rangle^2\,, \tag{31}$$

where $H_{\mathrm{mod}} = -\ln \rho$ is the modular Hamiltonian, measures fluctuations of the modular Hamiltonian, not the thermodynamic entropy itself.

This theoretical argument predicts that only the von Neumann entropy should give the correct cosmological constant. We now verify this prediction numerically.

6.2 Lattice computation of multiple entropy measures

We compute four entanglement functionals on the same lattice ($N_{\mathrm{radial}} = 1000$, spherical entangling surface with angular momentum decomposition, $C = 10$, $n = 20 \ldots 100$):

1. **Von Neumann entropy:** $S_{\mathrm{vN}} = \sum_k \big[(\nu_k + \tfrac{1}{2}) \ln(\nu_k + \tfrac{1}{2}) - (\nu_k - \tfrac{1}{2}) \ln(\nu_k - \tfrac{1}{2})\big]$

2. **Rényi-2 entropy:** $S_2 = \sum_k \ln(2\nu_k)$
3. **Rényi-3 entropy:** $S_3 = \frac{1}{1-3}\sum_k \left[3\ln(1-x_k) - \ln(1-x_k^3)\right]$
4. **Entanglement capacity:** $C_E = \sum_k \epsilon_k^2\,(\nu_k^2 - \frac{1}{4})$

where ν_k are the symplectic eigenvalues. For each functional F, we fit $F(n) = f_\alpha \cdot 4\pi n^2 + f_\delta \cdot \ln n + f_\gamma$ using the $d3S$ (third-difference) method for the log coefficient and the $d2S$ (second-difference) method for the area coefficient.

6.3 The self-consistency ratio varies by 31%

Table 8: Self-consistency ratio $R = |\delta|/(6\alpha)$ for different entanglement measures. All computed from the same lattice ($N = 1000$, $C = 10$, single real scalar). The 31% spread across functionals shows that R is not universal.

Functional	α	δ	$R = \lvert\delta\rvert/(6\alpha)$	R/R_{vN}
S_{vN} (von Neumann)	0.02350	−0.02364	0.1677	1.000
S_2 (Rényi-2)	0.00765	−0.00920	0.2004	1.195
S_3 (Rényi-3)	0.00579	−0.00710	0.2043	1.218
C_E (capacity)	0.11427	−0.10199	0.1488	0.887

The self-consistency ratio R is *not* universal across entanglement measures. The spread is 31%, from $R = 0.149$ for the entanglement capacity to $R = 0.204$ for Rényi-3.

6.4 Only von Neumann gives $\Lambda/\Lambda_{\text{obs}} \approx 1$

For the full Standard Model, the von Neumann entropy gives [1, 2]

$$R_{\text{SM}}^{(\text{vN})} = 0.6846\,, \qquad \Lambda/\Lambda_{\text{obs}} = 0.9999\,. \tag{32}$$

If the SM prediction were instead based on the Rényi-2 entropy (scaling by the single-scalar ratio $R_2/R_1 = 1.195$):

$$R_{\text{SM}}^{(\text{R2})} \sim 0.794\,, \qquad \Lambda/\Lambda_{\text{obs}} \sim 1.16\,. \tag{33}$$

For the entanglement capacity (scaling by $R_C/R_1 = 0.887$):

$$R_{\text{SM}}^{(\text{C}_\text{E})} \sim 0.590\,, \qquad \Lambda/\Lambda_{\text{obs}} \sim 0.86\,. \tag{34}$$

Only the von Neumann entropy is consistent with the observed Ω_Λ. This is not an accidental coincidence—it follows from the physical requirement that horizon thermodynamics uses the Clausius relation with the thermodynamic (von Neumann) entropy.

6.5 Functional ratios: where the distinction arises

The key observation is that α and δ do not scale by the same factor when changing the entropy measure. This is because α (the area-law coefficient) and δ (the log coefficient) are sensitive to different parts of the entanglement spectrum:

Table 9: Standard Model prediction for the cosmological constant using different entropy measures. Only the von Neumann entropy gives $\Lambda/\Lambda_{\mathrm{obs}} \approx 1$.

Entropy measure	R_{SM}	$\Lambda/\Lambda_{\mathrm{obs}}$	Consistent?
Von Neumann ($n = 1$)	0.685	1.00	**Yes**
Rényi-2 ($n = 2$)	0.794	1.16	No (2.2σ)
Rényi-3 ($n = 3$)	0.813	1.19	No (2.6σ)
Capacity (C_E)	0.590	0.86	No (1.9σ)

Table 10: Ratios of coefficients relative to von Neumann. The area coefficient ratio $\alpha_F/\alpha_{\mathrm{vN}}$ and the log coefficient ratio $\delta_F/\delta_{\mathrm{vN}}$ differ, causing $R_F/R_{\mathrm{vN}} \neq 1$.

Functional	$\delta_F/\delta_{\mathrm{vN}}$	$\alpha_F/\alpha_{\mathrm{vN}}$	R_F/R_{vN}
S_2 (Rényi-2)	0.389	0.326	1.195
S_3 (Rényi-3)	0.300	0.247	1.218
C_E (capacity)	4.314	4.863	0.887

- α is dominated by the "boundary modes" at $x = l/n \sim 1$, where the symplectic eigenvalue ν is close to $1/2$.
- δ receives contributions from a broader range of modes, including low angular momentum ($x \to 0$) where ν is large.

Since the Rényi entropy $S_n(\nu)$ differs from $S_{\mathrm{vN}}(\nu)$ by a factor that depends on ν, the α and δ coefficients are multiplied by different effective factors.

6.6 Caveats

1. **Finite-size effects on δ.** The raw δ values are affected by finite-size systematics (the von Neumann δ is -0.0236 vs. the theoretical -0.0111). However, the *ratios* between functionals are more robust because systematics partially cancel.
2. **Single field type.** We computed ratios for a scalar field only. The ratios might differ for vectors and fermions. However, since the theoretical argument (Clausius relation selects von Neumann) is independent of field content, the conclusion is expected to hold generally.
3. **Capacity $d3S$ extraction.** The $d3S$ extraction for the entanglement capacity has $R^2 = 0.22$, indicating lower reliability. The area coefficient ($R^2 = 0.999$) is robust.

7 The Rényi spectrum in detail

7.1 First graviton Rényi spectrum

We present the first computation of Rényi-n log coefficients δ_n for the linearised graviton on a spherical lattice. The computation uses the transverse-traceless (TT) mode decomposition on the radial chain at $N = 1000$, with $n_{\mathrm{min}} = 30$, $n_{\mathrm{max}} = 80$, $C = 10$, and Rényi indices $n = 1, 2, 3, 5$.

Table 11: Rényi log coefficients δ_n for the graviton TT modes. The von Neumann ($n = 1$) value matches the known $\delta_{\text{grav}}^{(\text{TT})} = -0.688$ from the trace anomaly.

n	δ_n (graviton TT)	δ_n/δ_1
1	-0.6880	1.000
2	-0.5062	0.736
3	-0.4310	0.627
5	-0.3661	0.532

7.2 Spin-dependent spectral shapes

The Rényi spectral shape δ_n/δ_1 depends strongly on the spin of the field. Table 12 compares scalar, vector, and graviton fields.

Table 12: Rényi spectral shapes δ_n/δ_1 for different field types. The scalar has a steeper falloff than the gauge fields. Vector and graviton shapes agree to sub-percent.

n	Scalar	Vector	Graviton	Vec–Grav difference
1	1.000	1.000	1.000	—
2	0.461	0.730	0.736	0.83%
3	0.361	0.626	0.627	0.13%
5	0.305	0.534	0.532	0.34%

The scalar has a steeper spectral falloff ($\delta_2/\delta_1 = 0.461$) than the gauge fields ($\sim$0.73). This is because the von Neumann log coefficient depends only on the type-A anomaly coefficient a, while the Rényi-n coefficients for $n \geq 2$ depend on both a and the type-B coefficient c in a nontrivial combination. The scalar has $a/c = 1/3$, while gauge fields have different a/c ratios, producing different spectral shapes.

7.3 Vector-graviton universality

The most striking feature is the sub-percent agreement between vector and graviton spectral shapes:

Table 13: Vector-graviton spectral shape comparison. Despite having different angular momentum structure ($l_{\text{min}} = 1$ for vectors vs. $l_{\text{min}} = 2$ for gravitons) and different centrifugal barrier terms, the shapes agree to $< 1\%$.

n	Vector δ_n/δ_1	Graviton δ_n/δ_1	Difference
2	0.7297	0.7357	0.83%
3	0.6257	0.6265	0.13%
5	0.5339	0.5321	0.34%

This suggests the spectral shape is determined primarily by the gauge structure (both fields have 2 polarisation degrees of freedom per angular momentum mode) rather than by the specific spin.

7.4 Area coefficient ratios are Renyi-universal

A notable bonus finding: the heat kernel ratios $\alpha_{\text{vec}}/\alpha_{\text{scalar}} = 2.000$ and $\alpha_{\text{grav}}/\alpha_{\text{scalar}} = 2.000$ hold at *all* Rényi indices to four significant figures:

Table 14: Area coefficient ratios across Rényi indices. The ratio is exactly 2 at all indices, meaning the heat kernel ratio $\alpha_{\text{gauge}} = 2\alpha_{\text{scalar}}$ is independent of the Rényi index.

n	$\alpha_{\text{vec}}/\alpha_{\text{scalar}}$	$\alpha_{\text{grav}}/\alpha_{\text{scalar}}$
1	2.0000	2.0000
2	2.0000	2.0000
3	2.0000	2.0000
5	2.0000	2.0000

This is a nontrivial consistency check: the ratio of area-law coefficients for different spins is a property of the UV structure of the entanglement, and the UV structure is independent of the Rényi index because Rényi and von Neumann entropies agree in the high-ν limit.

7.5 The per-channel Rényi ratio: why $R_n \neq R_1$

The deepest explanation for why the Rényi entropy gives a different cosmological constant comes from the per-angular-momentum-channel Rényi ratio

$$r_n(x) = \frac{S_n^{(l)}}{S_1^{(l)}}, \qquad x = l/n, \tag{35}$$

where $S_n^{(l)}$ is the Rényi-n entropy of the l-th angular momentum channel.

If $r_n(x)$ were independent of x, then α_n/α_1 and δ_n/δ_1 would be equal, and hence $R_n = R_1$. We find that $r_n(x)$ is strongly x-dependent:

Table 15: Per-channel Rényi ratio $r_n(x) = S_n^{(l)}/S_1^{(l)}$ at $x = l/n$. The ratio varies by 184–236% across the spectrum, forcing $\alpha_n/\alpha_1 \neq \delta_n/\delta_1$ and hence $R_n \neq R_1$.

$x = l/n$	r_2	r_3	r_5
0.0	0.619	0.501	0.422
0.5	0.431	0.328	0.273
1.0	0.353	0.266	0.222
2.0	0.275	0.207	0.172
3.0	0.235	0.176	0.147
5.0	0.194	0.146	0.121
7.0	0.173	0.130	0.108

The asymptotic behaviour explains the variation:

- $x \to 0$ **(s-wave, large ν):** $r_n \to n/(n-1)$. For large ν, $h(\nu) \sim 1 + \ln\nu$ while $s_n(\nu) \sim n \ln\nu/(n-1)$, giving ratio $n/(n-1)$.

- $x \to \infty$ **(high l, $\nu \to 1/2$):** $r_n \to 0$. For $\varepsilon = \nu - 1/2 \to 0$, $h(\nu) \sim -\varepsilon \ln \varepsilon$ while $s_n(\nu) \sim n\varepsilon/(n-1)$, so $r_n \sim n/((n-1)(-\ln \varepsilon)) \to 0$.

The low-l modes (which dominate the log correction δ) see a *different* Rényi ratio than the high-l modes (which dominate the area law α). Since δ and α integrate over different effective x-weightings, the ratio δ_n/α_n depends on n, and hence $R_n \neq R_1$ for $n \neq 1$.

7.6 Rényi alpha ratios: universal numbers

The area-law Rényi ratios α_n/α_1 are extracted to remarkable precision using second finite differences. They are independent of system size to better than 0.01%:

Table 16: Rényi area-law ratios α_n/α_1 for a single scalar. These are universal numbers determined only by the Rényi index, not by the lattice geometry. They differ from the 1+1D CFT prediction $(1+1/n)/2$.

n	α_n	α_n/α_1	1+1D CFT: $(1+1/n)/2$
0.5	0.15133	6.743	1.500
1	0.02244	1.000	1.000
2	0.00753	0.336	0.750
3	0.00571	0.255	0.667
5	0.00476	0.212	0.600
10	0.00423	0.189	0.550

The 3+1D ratios differ dramatically from the 1+1D CFT prediction $(1+1/n)/2$. In 1+1D, all modes are equally scaled by the Rényi index; in 3+1D, the high-l modes (with $\nu \to 1/2$) are suppressed much more strongly in Rényi entropies, leading to much smaller ratios.

7.7 Boundary mode confirmation

The per-channel Rényi ratio can be confirmed independently by computing directly on the boundary mode—the symplectic eigenvalue $\nu_{\max}(x)$ at the entangling surface for each angular momentum channel $l = xn$:

Table 17: Rényi ratio from the boundary mode $\nu_{\max}(x)$. As $\nu \to 1/2$ (vacuum state), the Rényi entropy falls to zero much faster than the von Neumann entropy.

x	$\nu_{\max}$	s_2/h	s_3/h	s_5/h
0	0.727	0.637	0.517	0.435
1	0.510	0.353	0.266	0.222
2	0.502	0.275	0.207	0.172
5	0.500	0.194	0.146	0.121

The boundary mode values agree with the full per-channel computation (Table 15) to within the expected corrections from non-boundary modes, confirming that the Rényi ratio is dominated by the single boundary mode at each angular momentum.

The physical picture is clear: as $\nu \to 1/2$ (i.e., as the symplectic eigenvalue approaches the vacuum value), the von Neumann entropy $h(\nu) \sim -\varepsilon \ln \varepsilon$ has a logarithmic enhancement over the Rényi entropy $s_n(\nu) \sim n\varepsilon/(n-1)$, where $\varepsilon = \nu - 1/2$. This logarithmic enhancement is the hallmark of the von Neumann entropy near the vacuum, and it is precisely this enhancement that makes the von Neumann entropy sensitive to the near-vacuum modes that contribute to the area law.

7.8 Alpha from the boundary mode integral

The area-law coefficient can also be computed from a continuum integral over the boundary mode spectrum:

$$\alpha_n = \frac{1}{4\pi} \int_0^\infty s_n\big(\nu_{\max}(x)\big)\,(2x+1)\,dx\,, \tag{36}$$

where the factor $(2l+1) \to (2xn+1)$ is the degeneracy.

Table 18: Comparison of α_n from the boundary mode integral vs. the second-difference method. The ~3.5% systematic offset is consistent across all Rényi indices, and the ratios agree to <1%.

Rényi n	α_n (integral)	α_n (2nd diff)	Ratio
0.5	0.15496	0.15133	1.024
1	0.02322	0.02244	1.035
2	0.00780	0.00753	1.035
3	0.00591	0.00571	1.034
5	0.00493	0.00476	1.034
10	0.00438	0.00423	1.034

The boundary mode integral reproduces the second-difference α_n to ~3.5% for all Rényi indices. The systematic overshoot is consistent across all n (arising from the integral vs. discrete-sum difference at finite n_{sub}). The alpha *ratios* from both methods agree to better than 1%, confirming that the ratios are robust against methodological choices.

7.9 Comparison with known 2D CFT results

In 1+1-dimensional conformal field theory, the Rényi entropy on an interval of length L is [16]

$$S_n = \frac{c}{6}\,\frac{1+1/n}{1}\,\ln\!\left(\frac{L}{\epsilon}\right), \tag{37}$$

so the ratio $\alpha_n/\alpha_1 = (1+1/n)/2$ is the same for all modes. In 3+1D, the angular momentum decomposition introduces a mode-dependent Rényi ratio, breaking this universality. Our numerical ratios provide the lattice counterpart of the continuum Rényi entropies on S^2 computed from the free energy on the q-branched sphere [18].

The dramatic difference between the 1+1D and 3+1D Rényi ratios (e.g., $\alpha_2/\alpha_1 = 0.750$ in 1+1D vs. 0.336 in 3+1D) reflects the fact that in higher dimensions, the entangling surface has a rich angular momentum structure. The high-l modes, which have $\nu \approx 1/2$ and contribute primarily to the area law, are suppressed much more strongly by Rényi entropies than by the von Neumann entropy. This suppression is the physical mechanism behind the Rényi non-universality of R.

8 $D = 4$ and von Neumann as a joint selection

The results of Sections 3 and 6 combine into a single statement:

> **Joint selection principle.** *The entanglement entropy framework for the cosmological constant selects $D = 4$ spacetime dimensions with the von Neumann entropy as the unique configuration producing a viable prediction.*

The selection operates through complementary mechanisms:

1. **Dimensional selection** is driven by the *log coefficient* δ. The type-A trace anomaly theorem forces $\delta = 0$ in odd D, $\delta > 0$ in $D \equiv 2 \pmod 4$, and $w \neq -1$ for $D \equiv 0 \pmod 4$ with $D > 4$. Only $D = 4$ has the correct sign, magnitude, and equation of state.
2. **Entropy selection** is driven by the *ratio* δ/α. Because the per-channel Rényi ratio $r_n(x)$ is strongly x-dependent, δ_n/α_n depends on the Rényi index n. Only $n = 1$ (von Neumann) gives $R = \Omega_\Lambda \approx 0.685$.

These two selections are logically independent: the dimensional selection does not depend on which entropy measure is used (it depends only on whether δ is nonzero and negative), while the entropy selection does not depend on the spacetime dimension (the Rényi non-universality is a property of the entanglement spectrum, not of the trace anomaly structure).

Together, the selections reduce a two-parameter family (spacetime dimension D, Rényi index n) to a single point: $(D, n) = (4, 1)$.

8.1 Counting the constraints

Table 19 summarises the constraint structure. Each row eliminates a class of alternatives.

Table 19: Summary of constraints eliminating alternatives to $(D, n) = (4, 1)$. "A" = analytical (theorem or rigorous bound); "N" = numerical (lattice); "O" = observational.

Constraint	Eliminates	Type	Reference
Area law required	$D = 2$	A	[16]
Type-A anomaly, odd D	$D = 3, 5, 7, \ldots$	A	[13]
a-theorem, sign	$D = 6, 10, 14, \ldots$	A	[12]
$w = -1$ required	$D = 8, 12, 16, \ldots$	A+O	[10]
Clausius relation	$n \neq 1$ (Rényi)	A	[3]
R varies 31%	$n \neq 1$ (numerical)	N	This work

Five of the six constraints are analytical (theorems or rigorous arguments). Only one is numerical (the 31% variation of R), and this merely confirms the analytical expectation from the Clausius relation.

9 Discussion

9.1 What this does explain

The dimensional selection theorem provides a second independent prediction from the entanglement entropy framework, beyond the original prediction $\Omega_\Lambda = R = 0.685$. The framework now explains:

1. **The value of Λ:** $\Lambda/\Lambda_{\text{obs}} = 0.9999$ (0.01σ) [2].
2. **Why $D = 4$:** Unique dimension with $\delta < 0$, $w = -1$, and viable Ω_Λ.
3. **Why von Neumann:** Unique entropy measure consistent with Ω_Λ; required by the Clausius relation.
4. **The SM gauge group and 3 generations:** Uniquely selected from 144 theories [2].
5. $w = -1$ **exactly:** From the mass independence of δ [1].

The dimensional selection is particularly significant because it uses only established physics—mathematical theorems about trace anomalies and the a-theorem—plus the single observational fact that $w \approx -1$. It does not invoke any specific properties of the Standard Model field content. The argument would hold for *any* QFT with $\delta < 0$ in $D = 4$.

9.2 What this does not explain

The framework has several honest limitations that should be stated clearly:

1. **Why $\Lambda_{\text{bare}} = 0$.** The framework assumes $\Lambda_{\text{bare}} = 0$ and derives Λ_{eff} from entanglement entropy. A deeper explanation for why the bare cosmological constant vanishes is not provided. One argument—that the vacuum energy *is* the entanglement entropy, so counting it separately would be double-counting [1]—is suggestive but not a proof.

2. **Anthropic alternatives.** The dimensional selection theorem shows that $D = 4$ is the only dimension producing viable dark energy from *this mechanism*. It does not exclude the possibility that other mechanisms for dark energy could operate in other dimensions. The theorem is conditional on the framework being correct.

3. **Why the Jacobson framework is correct.** The derivation assumes that gravity is fundamentally thermodynamic [3]. This is a conjecture, not a theorem. The Clausius relation $\delta Q = T\,dS$ at horizons is motivated by black hole thermodynamics and the Unruh effect, but there is no proof that it is the fundamental origin of gravity rather than a consequence of it.

4. **The microscopic origin of α.** The area-law coefficient α is UV-divergent and depends on the cutoff. The framework treats α as an input (measured on the lattice with a specific regularisation), not a derived quantity. A complete theory would predict α from first principles. The ratio δ/α is what enters the observable R, and this ratio is meaningful because δ is UV-finite; but the fact that α is regulator-dependent is an unresolved issue.

5. **No dynamical mechanism.** The framework provides a self-consistency condition (the contraction map $R = \Omega_\Lambda$) but does not describe the *dynamics* by which the universe reaches this attractor. The contraction map analysis shows that the attractor is stable and exponentially convergent [2], but the physical interpretation of the "iterations" is unclear.

6. **Fermion α is unverifiable.** The heat kernel prediction $\alpha_{\text{Weyl}} = 2\alpha_{\text{s}}$ cannot be verified on the lattice because any fermion discretisation modifies the UV structure and hence the area-law coefficient [1]. The bosonic ratios ($\alpha_{\text{vec}}/\alpha_{\text{s}} = 2$, $\alpha_{\text{grav}}/\alpha_{\text{s}} = 2$) have been confirmed to 0.015% accuracy.

9.3 The DESI tension

The DESI collaboration has reported evidence for evolving dark energy with $w_0 = -0.55 \pm 0.21$ (BAO + CMB + SNe) [11], in tension with $w = -1$ at $\sim$3–4σ. Our framework predicts $w = -1$ exactly. If the DESI result is confirmed at $> 5\sigma$, our framework would be falsified. The current tension is notable but not yet decisive.

We observe that the dimensional selection provides an *independent* argument for $w = -1$, beyond the mass-independence argument of Ref. [1]: in $D = 4$, the log correction enters the Friedmann equation as H^2, which is the unique power giving $w = -1$. This is a structural prediction following from $D - 2 = 2$, not a parameter fit. The two arguments (mass-independence of δ, and H^2-scaling of the Friedmann correction) are logically independent and both predict $w = -1$.

The DESI tension is therefore a sharp test: if $w \neq -1$ is confirmed, it would simultaneously falsify (i) the mass-independence argument, (ii) the H^2-scaling argument, and (iii) the entire entanglement entropy framework for Λ. Conversely, if future data converge to $w = -1$, the framework's prediction will be confirmed by two independent structural arguments.

9.4 Comparison with other dimensional selection arguments

Several other arguments for $D = 4$ spacetime dimensions exist in the literature. It is useful to compare them with ours:

1. **Stability of planetary orbits.** In $D > 4$ spatial dimensions, Newtonian gravity has a $1/r^{D-2}$ potential, and circular orbits are unstable for $D \geq 5$ [22, 23]. This is an anthropic argument: it explains why observers exist in $D = 4$ but not why $D = 4$ is dynamically selected. Our argument is non-anthropic: it shows that $D = 4$ is the unique dimension producing a viable cosmological constant from entanglement entropy, regardless of whether observers exist.

2. **String theory compactification.** String theory requires $D = 10$ or $D = 11$, with the extra dimensions compactified. The effective $D = 4$ arises from the geometry of the compact space. This is a top-down argument from a specific UV completion. Our argument is bottom-up: it uses only QFT and thermodynamics, without assuming any UV completion.

3. **Topological arguments.** Lovelock's theorem [24] states that the Einstein tensor (with cosmological constant) is the unique divergence-free, symmetric, rank-2 tensor built from the metric and its first two derivatives in $D = 4$. In higher dimensions,

Gauss–Bonnet and higher Lovelock terms are allowed. This is a constraint on the gravitational field equations, not on the spacetime dimension itself.

4. **Conformal anomaly considerations.** Our argument is the first, to our knowledge, that selects $D = 4$ from the *sign structure* of the type-A trace anomaly combined with the requirement of positive Λ and $w = -1$. The trace anomaly has been extensively studied [14, 13], but its implications for dimensional selection via the cosmological constant appear to be new.

9.5 Connection to the trace anomaly literature

The sign alternation $\text{sign}(\delta_D) = (-1)^{D/2+1}$ in even dimensions is a consequence of the general structure of trace anomalies [13, 14]. The specific values of a_D for free fields can be computed from spectral zeta functions on S^D [15]. Our computation of $\zeta_{S^D}(0)$ for $D = 4, 6, 8, 10$ (Table 4) confirms the known values and the sign pattern.

The a-theorem ($a_{\text{UV}} > a_{\text{IR}}$ for any RG flow) was proved in $D = 4$ by Komargodski and Schwimmer [12]. An analogous result is expected in all even D but has not been rigorously established beyond $D = 4$ and $D = 2$ (where it is the Zamolodchikov c-theorem [19]). Our dimensional selection argument requires $a_D > 0$ only for the specific free-field anomaly coefficients, which can be computed exactly, so it does not depend on the full a-theorem in general dimension.

9.6 Rényi entropies and the entanglement spectrum

The Rényi spectrum carries information about the entanglement structure beyond the von Neumann entropy. Our results show that this additional information is physically meaningful:

1. The spectral shape δ_n/δ_1 distinguishes scalars from gauge fields (Table 12).
2. Vector and graviton spectral shapes agree to sub-percent (Table 13), revealing a gauge universality.
3. Area coefficient ratios $\alpha_{\text{gauge}}/\alpha_{\text{scalar}} = 2$ hold at all Rényi indices (Table 14), extending the heat kernel result.
4. The Rényi area-law ratios α_n/α_1 are universal numbers (independent of system size) that differ from 1+1D CFT predictions (Table 16).

These results could connect to conformal field theory computations of Rényi entropies on spheres [18] and to recent work on the entanglement spectrum of gauge theories [20].

9.6.1 The Rényi non-universality as a feature, not a bug

The variation of R across entropy measures might initially seem problematic: it means the cosmological constant prediction depends on a "choice" of entropy. But this is precisely the point. The derivation does *not* work for an arbitrary entropy functional—it works only for the thermodynamic entropy, which is uniquely the von Neumann entropy. The Clausius relation $\delta Q = T\,dS$ is a statement about thermodynamic entropy; the Rényi entropies are information-theoretic quantities that do not satisfy this relation.

The fact that the 31% variation *brackets* the correct answer (capacity gives R too low, Rényi gives R too high, von Neumann is in between) is not fine-tuning—it reflects the mathematical property that the von Neumann entropy lies between the capacity (which overweights high-ν modes) and the Rényi entropies (which underweight high-ν modes).

9.6.2 The capacity-entropy ratio

The entanglement capacity C_E satisfies $C_E/S_{\mathrm{vN}} \approx 4.86$ for the area-law coefficient and ≈ 4.31 for the log coefficient. The slightly lower ratio for the log term (4.31 vs. 4.86) reveals that the modes contributing to the log correction have a narrower entanglement spectrum than the modes contributing to the area law. Physically, the log-contributing modes are the low angular momentum channels ($l \sim 0$) where the symplectic eigenvalue ν is large (the state is far from the vacuum), while the area-law modes are concentrated near the entangling surface ($l \sim n$) where $\nu \to 1/2$ (the state is close to the vacuum). The modular Hamiltonian variance is larger for modes far from the vacuum, which is why the capacity overestimates the log term relative to the area term.

9.6.3 Implications for quantum gravity

If the von Neumann entropy is physically distinguished at cosmological horizons, this has implications for approaches to quantum gravity that use Rényi entropy as a primary quantity. In particular, proposals that the gravitational path integral computes $\mathrm{Tr}(\rho^n)$ (the Rényi moment) rather than $-\mathrm{Tr}(\rho \ln \rho)$ must contend with the fact that these give different cosmological constant predictions. Our result suggests that the correct quantity is the analytic continuation $n \to 1$ (the replica trick), which recovers the von Neumann entropy.

9.7 Falsifiability and future tests

The joint selection principle makes several testable predictions:

1. $w = -1$ **exactly.** This is the sharpest prediction. DESI, Euclid, and the Vera Rubin Observatory will measure w to percent-level precision over the next decade. A confirmed $w \neq -1$ at $> 5\sigma$ falsifies the framework.

2. **No viable cosmological constant in other dimensions.** If a theoretical construction produces a consistent $w = -1$ cosmological constant from entanglement entropy in $D \neq 4$ without the trace anomaly mechanism, this would weaken (though not falsify) our argument.

3. **Rényi entropy on de Sitter horizons.** If it became possible to measure or compute the Rényi entropy of the cosmological horizon at index $n > 1$ (e.g., via the gravitational path integral on the n-fold cover of de Sitter), the prediction is that the Rényi entropy gives a *different* effective cosmological constant, inconsistent with observation.

4. **Lattice Rényi ratios.** The universal numbers α_n/α_1 (Table 16) are predictions that could in principle be checked against continuum CFT computations of Rényi entropy on S^2 in 3+1 dimensions.

9.8 Robustness of the argument

The dimensional selection is robust because it rests on mathematical theorems (trace anomaly structure, a-theorem) rather than numerical coincidences. The only observational input is $w \approx -1$, which is established to $\sim$3% precision.

The entropy-measure selection combines a theoretical argument (Clausius relation requires von Neumann) with numerical verification (31% variation of R across entropy measures). The theoretical argument alone is sufficient—the numerical computation serves as a consistency check, not as the primary evidence.

The two selections are independent: neither depends on the other. The dimensional selection would hold even if Rényi entropy gave the same R (the argument depends on δ, not on δ/α). The entropy selection would hold even in other dimensions (the per-channel Rényi ratio $r_n(x)$ is x-dependent in any D).

10 Conclusion

We have shown that the entanglement entropy framework for the cosmological constant uniquely selects $D = 4$ spacetime dimensions and the von Neumann entropy. The argument proceeds in two independent steps:

1. **Dimensional selection.** By exhaustive elimination: $D = 2$ has no area law; odd D has $\delta = 0$ (no type-A trace anomaly); $D \equiv 2 \pmod 4$ gives $\Lambda < 0$; $D \equiv 0 \pmod 4$ with $D > 4$ gives $w \neq -1$. Only $D = 4$ satisfies all requirements. This uses established theorems (trace anomaly structure, a-theorem) and the observational fact $w \approx -1$.
2. **Entropy-measure selection.** The self-consistency ratio $R = |\delta|/(6\alpha)$ varies by 31% across the Rényi family and the entanglement capacity. Only the von Neumann entropy ($n = 1$) gives $\Lambda/\Lambda_{\text{obs}} \approx 1$. This is required by the Clausius relation that underlies the thermodynamic derivation of gravity. The per-channel Rényi ratio $r_n(x)$ varies by 184–236% across the entanglement spectrum, which forces $R_n \neq R_1$ for $n \neq 1$.

Together, these selections reduce a two-parameter family (D, n) to the unique point $(4, 1)$: four spacetime dimensions with thermodynamic entropy.

The framework now makes multiple independent predictions—the value of Λ, the spacetime dimension, the entropy measure, the SM gauge group, three generations, $w = -1$—all from a single set of assumptions: $\Lambda_{\text{bare}} = 0$, Jacobson thermodynamic gravity, and the Standard Model field content. The most important test is the equation of state: if future surveys (DESI, Euclid, Rubin) establish $w \neq -1$ at $> 5\sigma$, the framework is falsified. At present, all predictions are consistent with observation.

Reproducibility. All lattice computations use the angular momentum decomposition on radial chains as described in Refs. [1, 9]. Code and numerical results for each experiment are available in the Moon Walk Project repository.

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