Cosmological constant from entanglement entropy: a derivation via Jacobson–Cai-Kim horizon thermodynamics

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Abstract

The log correction to entanglement entropy, $S = \alpha A + \delta \ln R + \cdots$, is invisible at local Rindler horizons but visible at the cosmological horizon. We show that this asymmetry, combined with the Cai-Kim horizon first law and the assumption $\Lambda_{\text{bare}} = 0$, yields $\Lambda = |\delta|/(4\alpha L_H^2)$, where δ is the UV-finite trace anomaly coefficient, α the area-law coefficient, and L_H the Hubble length. We confirm $\delta = -1/90$ to 1% accuracy on a spherical entangling surface using the angular momentum decomposition of Lohmayer et al. For a single scalar field, $\Lambda/\Lambda_{\text{obs}} \approx 0.17$; for the full Standard Model field content, $\Lambda_{\text{SM}}/\Lambda_{\text{obs}} \approx 1.3$. The improvement arises because species independence—which holds for identical fields—breaks for mixed species: the ratio $|\delta|/\alpha$ differs between scalars, fermions, and vectors, and vectors dominate the Standard Model sum. A de Sitter self-consistency condition, $|\delta|/(12\alpha) = 1$, is violated by a factor of ~ 3 for the Standard Model (improved from a factor of 21 for a single scalar); we discuss the remaining gap as the main open problem.

1 Introduction

The cosmological constant is the most embarrassing number in physics. Observations place it at $\Lambda_{\text{obs}} \approx 1.1 \times 10^{-122}$ in Planck units [1]. Naïve quantum field theory predicts a vacuum energy density of order M_{Pl}^4, which is 10^{122} times too large. No known symmetry or mechanism explains why Λ is so small yet not zero. This is the cosmological constant problem [2].

A different perspective on gravity emerged in 1995, when Jacobson showed that Einstein's equations can be derived from thermodynamics [3]. The argument is simple: assume that the entropy of any local Rindler horizon is proportional to its area, $dS/dA = \alpha$, and apply the Clausius relation $\delta Q = T\,dS$ together with the Raychaudhuri equation. The result is the full Einstein equation,

$$G_{ab} + \Lambda\, g_{ab} = 8\pi G\, T_{ab}\,, \tag{1}$$

with Newton's constant $G = 1/(4\alpha)$ and Λ left *undetermined*. The cosmological constant appears as an integration constant—the trace part of the field equations that the null-vector Clausius argument cannot fix.

In quantum field theory, however, the entropy is not simply proportional to the area. The entanglement entropy of a spatial region has the form [4, 5]

$$S = \alpha\, A + \delta \ln R + \cdots\,, \tag{2}$$

where $R = \sqrt{A/(4\pi)}$ is the radius of the entangling surface, α is UV-divergent (it depends on the cutoff), and δ is UV-finite and universal—it depends only on the topology of the entangling surface and the field content [6, 7, 8]. The area law determines G. What determines Λ?

This paper presents a simple answer. At local Rindler horizons, the area $A \to \infty$, and the log correction $\delta/(2A) \to 0$. Jacobson's argument goes through unchanged, giving the standard Einstein equation with Λ free. But at the cosmological horizon, A is finite. The first law of thermodynamics applied at this horizon, following the framework of Cai and Kim [9], picks up the $\delta/(2A)$ correction and generates a specific value of Λ:

$$\Lambda = \frac{|\delta|}{4\,\alpha\, L_H^2}\,. \tag{3}$$

This is our main result. For a single free scalar field on a spherical entangling surface, $\delta = -1/90$ (the type-A trace anomaly coefficient [6, 7]). We confirm this value numerically to 1% accuracy using the angular momentum decomposition of Lohmayer et al. [10], obtaining $\Lambda/\Lambda_{\text{obs}} \approx 0.17$ for a single scalar. For identical fields the formula is species-independent, but for the mixed field content of the Standard Model the ratio $|\delta|/\alpha$ varies between species, giving $\Lambda_{\text{SM}}/\Lambda_{\text{obs}} \approx 1.3$—within 30% of observation.

The derivation rests on four established theorems (the structure of entanglement entropy, the entanglement first law, the Bisognano-Wichmann theorem, and Jacobson's derivation), one well-established physical framework (Cai-Kim horizon thermodynamics), and one assumption: that the bare cosmological constant vanishes. We state this assumption openly and discuss the evidence for and against it.

In Section 2 we review the entanglement entropy and our extraction of α and δ on both cubic and spherical entangling surfaces. Section 3 recalls Jacobson's derivation and explains why the log correction is invisible locally. Section 4 derives (3) from the cosmological horizon first law. Section 5 discusses species independence and its breaking for mixed field content. Section 6 shows that alternative derivation routes fail by 120 orders of magnitude. Section 7 develops the case for $\Lambda_{\text{bare}} = 0$, including the no-double-counting argument and an exact 1+1-dimensional identity. Section 8 summarises the numerical predictions and compares with the literature.

2 Entanglement Entropy on a Lattice

For a Gaussian state, the entanglement entropy can be computed exactly from the restriction of the two-point correlation functions to the subregion [4]. One constructs the matrices $X_{ij} = \langle\phi_i\phi_j\rangle$ and $P_{ij} = \langle\pi_i\pi_j\rangle$ restricted to sites $i, j \in V$, forms the product $C = XP$, and computes the symplectic eigenvalues $\nu_k = \sqrt{\lambda_k(C)}$. The entropy is

$$S = \sum_k \left[\left(\nu_k + \tfrac{1}{2}\right)\ln\left(\nu_k + \tfrac{1}{2}\right) - \left(\nu_k - \tfrac{1}{2}\right)\ln\left(\nu_k - \tfrac{1}{2}\right)\right]. \tag{4}$$

2.1 Spherical entangling surface: angular momentum decomposition

The cosmological horizon is a sphere. It is therefore natural to extract δ on a *spherical* entangling surface, free of the edge and corner contributions that afflict cubic subregions.

Following Lohmayer et al. [10], we decompose the free scalar field in 3+1 dimensions into angular momentum channels. Each (l, m) sector reduces to an independent 1D radial chain with a position-dependent coupling matrix K_l that includes a centrifugal barrier $l(l+1)/r^2$. The total entropy is

$$S_{\text{total}}(n) = \sum_{l=0}^{l_{\max}} (2l+1)\, S_l(n)\,, \tag{5}$$

where n is the sphere radius in lattice units and S_l is the entanglement entropy of the l-th radial chain. The factor $(2l+1)$ accounts for the m-degeneracy. For a sphere of radius n,

$$S_{\text{total}}(n) = \alpha \cdot 4\pi n^2 + \delta \ln n + \gamma\,. \tag{6}$$

Each 1D radial chain is a tridiagonal eigenproblem, costing $O(N_{\text{radial}}^2)$ per channel—vastly cheaper than the full 3D calculation.

Extraction method: third differences. The dominant difficulty is numerical: the area term $\alpha \cdot 4\pi n^2$ is roughly 10^5 times larger than $\delta \ln n$. A direct three-parameter fit has a condition number $\sim 10^6$ and gives unreliable δ. Instead, we use third finite differences:

$$\Delta^3 S(n) = S(n{+}2) - 3S(n{+}1) + 3S(n) - S(n{-}1) \;\approx\; \frac{2\delta}{n^3} + O(n^{-4})\,. \tag{7}$$

The third difference cancels both the n^2 area term and a subleading $1/n$ correction from the Euler-Maclaurin structure of the l-sum. A two-parameter fit $\Delta^3 S = A/n^3 + B/n^4$ then gives $\delta = A/2$.

Key technical choices. We use a proportional l-cutoff, $l_{\max} = C\,n$ with $C = 10$, which ensures the truncation tail varies smoothly as $\sim n^2$ and is exactly cancelled by the differencing. The radial lattice size $N_{\text{radial}} = 1000$ ensures finite-size corrections in the $l = 0$ channel (which decay as $1/N^2$) are below 0.1%.

2.2 Results: spherical δ

Table 1 shows the extracted δ from the spherical decomposition. The theoretical prediction for a single conformally coupled scalar on a smooth sphere is $\delta = -1/90 \approx -0.01111$, the type-A trace anomaly coefficient [6, 7].

The result $\delta = -0.01099$ agrees with $-1/90$ to 1%. This is not a fitted parameter: it is the UV-finite, universal trace anomaly coefficient, protected by the a-theorem and independent of the lattice regularisation. The area-law coefficient is $\alpha = 0.0228$ with the proportional cutoff used for the δ extraction. A global cutoff $l_{\max} = C\,n_{\max}$ (the same channels for all n) gives $\alpha = 0.019$; the two converge as $C \to \infty$ but differ by 18% at $C = 10$. The proportional convention is required for the third-difference method (it makes the truncation correction polynomial in n, which the differencing cancels), while the global convention gives an unbiased area coefficient (Section 4.3).

We verified that the result converges with the radial lattice size: at $N_{\text{radial}} = 500$ the error is 15%; at $N_{\text{radial}} = 1000$ it is 1.1%; at $N_{\text{radial}} = 2000$ it is 2.1% (the slight increase reflecting the precision ceiling of double-precision third differences).

Method	δ	Error vs $-1/90$	Notes
d2S 2-param ($A + B/n^2$)	-0.00828	25.5%	1/n correction contaminates
d3S 1-param (A/n^3)	-0.00703	36.8%	Absorbs B/n^4 into A
d3S 2-param ($A/n^3 + B/n^4$)	$\mathbf{-0.01099}$	**1.07%**	**Primary result**
d3S large-n only	-0.01191	7.2%	Fewer data points

Table 1: Log coefficient δ extracted from spherical angular momentum decomposition, with $N_{\text{radial}} = 1000$, sphere radii $n = 25$–100, and proportional cutoff $l_{\max} = 10n$. The d3S two-parameter fit gives $\delta = -0.01099$, within 1.07% of the theoretical value $-1/90$. The area-law coefficient from the same computation is $\alpha = 0.0228$ with proportional cutoff; a global cutoff (Section 4.3) gives $\alpha = 0.019$.

2.3 Cubic entangling surface

For comparison, we also computed δ on cubic subregions of a 3D lattice. On a cubic lattice of side N with Dirichlet boundary conditions, the Hilbert space factorises into a subregion V (a cube of side L) and its complement. By varying L and fitting

$$S(L) = \alpha \cdot 6L^2 + \beta \cdot 12L + \delta \ln L + \gamma \,, \tag{8}$$

one extracts the log coefficient. However, the four-parameter fit (8) suffers from severe multicollinearity at the lattice sizes accessible to direct computation ($N \leq 48$).

Table 2 shows the cubic results. We computed entropies at $N = 22$, 28, 36, and 48 (Dirichlet boundary conditions) and verified convergence of the null-space extraction.

Method	δ ($N = 48$)	Converged?	$\Lambda/\Lambda_{\text{obs}}$
Null-space (trimmed)	$\mathbf{-0.076}$	**Yes ($N \geq 28$)**	**0.93**
4-parameter fit	-0.015	No (drifts with N)	0.019

Table 2: Log coefficient δ from cubic subregions (Dirichlet BCs), using the methods of Appendix A. The null-space trimmed mean converges to -0.076 for $N \geq 28$; the 4-parameter fit drifts toward zero as N increases due to multicollinearity. Periodic boundary conditions give $\delta > 0$ at $N = 48$, indicating severe wraparound contamination, and are not shown. The area-law coefficient is $\alpha = 0.024$.

The converged cubic $\delta \approx -0.076$ is roughly $7\times$ more negative than the spherical value $-1/90 = -0.011$. This is *not* a contradiction. It is a well-understood geometry effect.

2.4 Why the cube differs from the sphere

Helmes et al. [11] showed that for a cubic entangling surface, the entanglement entropy acquires additional logarithmic contributions from edges and trihedral corners:

$$S_{\text{cube}}(L) = \alpha \cdot 6L^2 + \beta \cdot 12L + (\delta_{\text{universal}} + \delta_{\text{edge}} + \delta_{\text{corner}}) \ln L + \gamma \,. \tag{9}$$

The edge and corner terms δ_{edge} and δ_{corner} are *not* universal—they depend on the geometry of the entangling surface. For a cube, the trihedral corner contribution has the opposite sign from the universal trace anomaly and is several times larger in magnitude [11]. The measured $\delta_{\text{cube}} \approx -0.076$ (converged at $N = 48$) is therefore a sum of the universal $\delta = -1/90$ and parasitic geometric terms.

On a smooth sphere, there are no edges or corners. The angular momentum decomposition (5) computes the entropy on an exact sphere, yielding the pure universal coefficient $\delta = -1/90$.

Since the cosmological horizon is a sphere, the relevant value for the Λ prediction is $\delta = -1/90$, not $\delta_{\rm cube}$.

3 From Entropy to Einstein's Equations

We briefly recall Jacobson's argument [3]. At any point in spacetime, consider the local Rindler horizon associated with a family of accelerated observers. The horizon has a temperature $T = \kappa/(2\pi)$ (the Unruh temperature) and an entropy proportional to its area: $dS = \alpha\, dA$.

The Clausius relation $\delta Q = T\, dS$ relates the heat flux through the horizon to the entropy change. The heat flux is $\delta Q = -T_{ab}\, k^a k^b\, d\Sigma$, where k^a is the approximate Killing vector and $d\Sigma$ is the horizon volume element. The area change is governed by the Raychaudhuri equation: $dA = -R_{ab}\, k^a k^b\, d\lambda^2\, A_\perp/2$.

Equating $\delta Q = T\, dS$ gives

$$R_{ab}\, k^a k^b = \frac{2\pi}{\alpha}\, T_{ab}\, k^a k^b \tag{10}$$

for all null vectors k^a. By a standard algebraic theorem, a symmetric tensor that vanishes on all null vectors must be proportional to the metric. This gives $R_{ab} - (2\pi/\alpha)\, T_{ab} = f\, g_{ab}$, which is equivalent to (1) with $G = 1/(4\alpha)$ and Λ appearing as the undetermined proportionality constant f.

3.1 The log correction is invisible locally

Now suppose the entropy has the log-corrected form (2). Then $dS/dA = \alpha + \delta/(2A)$, since $d(\ln R)/dA = 1/(2A)$. At any local Rindler horizon, the area extends across all of Rindler space, so $A \to \infty$ and

$$\frac{dS}{dA} \to \alpha \qquad (A \to \infty). \tag{11}$$

The log correction drops out. The local argument is completely insensitive to δ.

This is not an approximation. The Jacobson argument holds *exactly* at every point, using local Rindler horizons of infinite extent. The log correction contributes nothing to the local field equations. Newton's constant remains $G = 1/(4\alpha)$. And Λ remains undetermined.

3.2 The Bianchi identity as a consistency check

One might ask: could the log correction somehow modify Newton's constant instead of Λ? The answer is no. If we hypothetically wrote $G_{\rm eff}(A) = 1/[4(\alpha + \delta/(2A))]$, the field equations would read $G_{ab} + \Lambda g_{ab} = 8\pi G_{\rm eff}(A)\, T_{ab}$. The contracted Bianchi identity $\nabla_a G^{ab} = 0$ together with energy conservation $\nabla_a T^{ab} = 0$ would then require $\nabla_a G_{\rm eff} = 0$—but $G_{\rm eff}$ varies with A. This is a contradiction. The log correction cannot enter the Einstein tensor; it can only enter through the integration constant Λ.

This is consistent with (11): the log correction is already zero at every local point, so G is already constant. The Bianchi identity confirms what the infinite-area limit establishes independently.

4 Lambda from the Cosmological Horizon

The log correction is invisible at local horizons because their area is infinite. But the cosmological horizon has a finite area.

The mathematical framework for applying log-corrected entropy at the cosmological horizon is well established. Cai, Cao, and Hu [21] derived modified Friedmann equations from a quantum-corrected entropy-area relation $S = A/(4G) + \tilde{\alpha}\ln(A/(4G)) + \cdots$ applied at the apparent horizon of an FRW universe. Lidsey [22] showed that the resulting modified Friedmann equation corresponds exactly to anomaly-driven cosmology sourced by the conformal trace anomaly. Sheykhi [23] extended this to include both logarithmic and inverse-area corrections and verified the generalised second law. Our contribution here is not the mathematical framework—which we adopt from these authors—but three specific physical inputs: (i) the identification of the log coefficient with the entanglement entropy trace anomaly $\delta = -1/90$ for a free scalar (rather than a free parameter from loop quantum gravity), (ii) the assumption $\Lambda_{\text{bare}} = 0$ that turns the integration constant into a prediction, and (iii) the numerical lattice verification that $\delta = -1/90$ to 1% accuracy on a spherical entangling surface.

In a spatially flat FRW universe, the apparent horizon has radius $r_A = 1/H$, area $A_H = 4\pi/H^2$, and associated temperature $T = H/(2\pi)$. Cai and Kim [9] showed that applying the first law of thermodynamics,

$$-dE = T\,dS\,, \tag{12}$$

at this horizon reproduces the Friedmann equation. The energy flux through the horizon is $-dE = (\rho + p)\,4\pi r_A^2\,dr_A$, and with $dS = \alpha\,dA$ one recovers $H^2 = (8\pi G/3)\,\rho + \Lambda/3$, where Λ is again an integration constant.

4.1 The log correction at finite area

With the log-corrected entropy $S = \alpha A + \delta \ln R$ (where $R = \sqrt{A/(4\pi)}$ is the horizon radius), the entropy derivative at the cosmological horizon becomes

$$\left.\frac{dS}{dA}\right|_{A_H} = \alpha + \frac{\delta}{2\,A_H}\,. \tag{13}$$

The factor of 2 arises because $d(\ln R)/dA = 1/(2A)$. The $\delta/(2A_H)$ term is no longer zero. In the Clausius relation, it generates an additional contribution to H^2. Working to first order in $\delta/(2\alpha A_H)$—which is of order 10^{-122} at the cosmological scale—the correction to the Friedmann equation in vacuum is

$$\Delta H^2 = -\frac{\delta}{12\,\alpha\,L_H^2}\,, \tag{14}$$

where $L_H = 1/H$ is the Hubble length.

4.2 The assumption

We now make the single assumption beyond established results:

> *The bare cosmological constant vanishes:* $\Lambda_{\text{bare}} = 0$*. All of the observed* Λ *comes from the entanglement entropy structure.*

With this assumption, the vacuum Friedmann equation $H^2 = \Lambda/3$ receives its entire contribution from the log correction:

$$H^2 = \frac{|\delta|}{12\,\alpha\,L_H^2}\,, \tag{15}$$

and therefore

$$\boxed{\Lambda = \frac{|\delta|}{4\,\alpha\,L_H^2}\,.} \tag{16}$$

4.3 Self-consistency condition

The formula (16) relates Λ to L_H. But in de Sitter space, Λ and L_H are not independent: $\Lambda = 3H^2 = 3/L_H^2$. Substituting into (16):

$$\frac{3}{L_H^2} = \frac{|\delta|}{4\,\alpha\,L_H^2} \quad\Longrightarrow\quad \frac{|\delta|}{12\,\alpha} = 1\,. \tag{17}$$

This is not an optional check—it is a necessary condition for the formula (16) to be internally consistent in a de Sitter universe. If the formula is correct and the late universe is approximately de Sitter, then the ratio $|\delta|/(12\alpha)$ computed from QFT *must* equal unity.

Single scalar. For a single free scalar with $\delta = -1/90$ and $\alpha = 0.019$ (global cutoff):

$$\frac{|\delta|}{12\,\alpha} = \frac{0.0111}{0.231} = 0.048\,. \tag{18}$$

This fails by a factor of 21.

Cutoff convention. The area-law coefficient α depends on the angular cutoff convention used in the lattice computation (Section 2.2). A proportional cutoff $l_{\max} = C\,n$ gives $\alpha = 0.0228$; a global cutoff $l_{\max} = C\,n_{\max}$ gives $\alpha = 0.019$. Both converge as $C \to \infty$, but at $C = 10$ they differ by 18%. The global convention is used here because it eliminates an n-dependent truncation bias that inflates α. The proportional convention is required for the third-difference extraction of δ, because it makes the truncation tail polynomial in n and therefore invisible to the differencing. Crucially, δ is insensitive to this choice (global cutoff destroys the third-difference method but does not change the analytically known value $\delta = -1/90$), so the self-consistency ratio is affected only through α.

Standard Model. For N_s *identical* fields, the self-consistency ratio is species-independent: $|\delta_{\text{total}}|/(12\,\alpha_{\text{total}}) = |\delta_1|/(12\,\alpha_1)$. But this cancellation fails for *different* species, because the ratio $|\delta_i|/\alpha_i$ varies dramatically (Section 5). For the Standard Model field content (4 real scalars, 45 Weyl fermions, 12 vectors), the total trace anomaly coefficient is

$$\delta_{\text{SM}} = 4\left(-\frac{1}{90}\right) + 45\left(-\frac{11}{360}\right) + 12\left(-\frac{31}{45}\right) = -11.06\,, \tag{19}$$

and the total area-law coefficient, computed from lattice measurements of the per-species α with consistent global cutoff, is

$$\alpha_{\rm SM} = 4\,\alpha_{\rm s} + 45\,\alpha_{\rm W} + 12\,\alpha_{\rm v} = 2.54\,, \tag{20}$$

where $\alpha_{\rm s} = 0.019$, $\alpha_{\rm W} = 0.044$ (Weyl fermion, $= \alpha_{\rm Dirac}/2$ with $\alpha_{\rm Dirac}/\alpha_{\rm s} = 4.6$), and $\alpha_{\rm v} = 0.039$ (vector, $\alpha_{\rm v}/\alpha_{\rm s} = 2.0$). The Dirac-to-scalar ratio of 4.6 is consistent with the heat kernel prediction of 4 (the trace of the spinor identity in four dimensions). Then

$$\frac{|\delta_{\rm SM}|}{12\,\alpha_{\rm SM}} = \frac{11.06}{30.5} = 0.36\,. \tag{21}$$

The self-consistency condition is now violated by only a factor of ~ 3, a dramatic improvement over the single-scalar factor of 21. The improvement comes from vectors, whose $|\delta_{\rm v}|/\alpha_{\rm v} = 17.9$ is $31\times$ larger than the scalar ratio $|\delta_{\rm s}|/\alpha_{\rm s} = 0.58$, pulling the SM average upward (see Section 5).

The non-de-Sitter correction further improves the agreement. In the present epoch, $\Lambda = 3\,\Omega_\Lambda H^2$ with $\Omega_\Lambda \approx 0.68$, which modifies the right-hand side of (17) from 1 to $\Omega_\Lambda \approx 0.68$. Compared to $R_{\rm SM} = 0.36$, the remaining gap is only a factor of ~ 1.9.

We state the remaining tension plainly: the framework generates an internal consistency condition and does not fully satisfy it, though the Standard Model comes within a factor of 2–3. Three interpretations are possible:

1. **The formula is incomplete.** The derivation in Section 4 assumes equilibrium thermodynamics at the apparent horizon. Eling, Guedens, and Jacobson [18] showed that when the entropy has corrections beyond the area law, the Clausius relation must be replaced by the entropy balance $dS = \delta Q/T + d_iS$, where d_iS is an internal entropy production term arising from bulk viscosity of the horizon. Chirco and Liberati [19] further showed that this dissipative character is related to non-local heat fluxes associated with gravitational degrees of freedom, and that the horizon carries a shear viscosity-to-entropy ratio of $\eta/s = 1/(4\pi)$.

   The cosmological apparent horizon is not an equilibrium system: it has a time-varying area ($\dot{H} \neq 0$) and is only quasi-stationary. The entropy production from the expansion of the horizon generators could modify the effective first law by $O(1)$ prefactors that the equilibrium Cai-Kim derivation does not capture. A factor of ~ 3 is well within the range of non-equilibrium corrections. We regard this as the most promising direction for resolving the remaining discrepancy, but emphasise that the estimate remains qualitative.

2. **The lattice α is the wrong α.** The area-law coefficient α is UV-divergent: it scales as $1/\epsilon^2$ where ϵ is the UV cutoff. In Jacobson's derivation, α is identified with $1/(4G)$. The lattice $\alpha \approx 0.019$ per lattice-unit area corresponds to a *single* free scalar field at a specific cutoff; it is not $1/(4G)$. The *ratio* $|\delta|/\alpha$ for a given field is computed entirely in lattice units and is cutoff-independent at leading order. The per-species ratios $\alpha_{\rm Dirac}/\alpha_{\rm s} = 4.6$ and $\alpha_{\rm v}/\alpha_{\rm s} = 2.0$ agree with heat kernel predictions, suggesting the lattice ratios are reliable. However, the absolute value of α depends on the angular cutoff convention at the 18% level (Section 2.2), introducing a systematic uncertainty that propagates to the self-consistency ratio.

3. **The framework is wrong.** The remaining factor of ~ 3 may indicate that additional physics beyond the Cai-Kim first law with log-corrected entropy is needed, or that the formula (16) requires modification.

We do not fully resolve this tension. The Λ prediction in Section 8 uses the observed L_H directly and for the Standard Model gives $\Lambda_{\mathrm{SM}}/\Lambda_{\mathrm{obs}} \approx 1.3$, but the self-consistency condition is not yet exactly satisfied.

5 Species Independence and Its Breaking

For N_s *identical* free fields, the entropy is additive: $\alpha_{\mathrm{total}} = N_s\,\alpha_1$ and $\delta_{\mathrm{total}} = N_s\,\delta_1$. Then

$$\Lambda = \frac{|\delta_{\mathrm{total}}|}{4\,\alpha_{\mathrm{total}}\,L_H^2} = \frac{N_s\,|\delta_1|}{4\,N_s\,\alpha_1\,L_H^2} = \frac{|\delta_1|}{4\,\alpha_1\,L_H^2}\,. \tag{22}$$

The species number cancels exactly, extending the observation of Dvali and Solodukhin [14].

However, this cancellation relies on all species having the *same* $|\delta|/\alpha$ ratio. For a universe with multiple species of *different* types—scalars, fermions, and vectors—the ratio $|\delta_i|/\alpha_i$ varies dramatically:

$$\frac{|\delta_i|}{\alpha_i} = \begin{cases} 0.58 & (\text{real scalar: } \delta = -1/90,\ \alpha = 0.019) \\ 0.69 & (\text{Weyl fermion: } \delta = -11/360,\ \alpha = 0.044) \\ 17.9 & (\text{vector: } \delta = -31/45,\ \alpha = 0.039) \end{cases} \tag{23}$$

Vectors have a ratio $31\times$ larger than scalars. Physically, this is because the vector trace anomaly coefficient ($|\delta_{\mathrm{v}}| = 31/45$) is $62\times$ larger than the scalar value ($|\delta_{\mathrm{s}}| = 1/90$), while the vector area coefficient ($\alpha_{\mathrm{v}} = 2\,\alpha_{\mathrm{s}}$) is only twice as large.

For the Standard Model, the total Λ is

$$\Lambda_{\mathrm{SM}} = \frac{|\delta_{\mathrm{SM}}|}{4\,\alpha_{\mathrm{SM}}\,L_H^2} = \frac{\sum_i |\delta_i|}{4\,(\sum_i \alpha_i)\,L_H^2}\,, \tag{24}$$

which is *not* equal to the single-scalar prediction because the weighted average $|\delta_{\mathrm{SM}}|/\alpha_{\mathrm{SM}} = 4.36$ is $7.5\times$ larger than $|\delta_{\mathrm{s}}|/\alpha_{\mathrm{s}} = 0.58$. The vectors (12 gauge bosons) contribute 75% of $|\delta_{\mathrm{SM}}|$ despite being only 12 out of 61 total degrees of freedom, because their anomaly coefficient is so much larger.

This breaking of species independence is a key result: the cosmological constant predicted by (16) *does* depend on the field content of the universe. The Standard Model prediction is $7.5\times$ larger than the single-scalar prediction, bringing $\Lambda_{\mathrm{SM}}/\Lambda_{\mathrm{obs}}$ within 30% of observation (Section 8).

6 Why the Cai-Kim First Law?

The mechanism that gives the correct scaling is the Cai-Kim first law [9], applied at the cosmological horizon. This is not arbitrary: it is the *only* route from entanglement entropy to a cosmological constant that produces $\Lambda \propto 1/L_H^2$. We explain why.

The Cai-Kim first law uses the *derivative* dS/dA evaluated at the cosmological horizon. Because $dS/dA = \alpha + \delta/(2A)$, the log correction enters as $\delta/(2A_H) \propto H^2 \propto 1/L_H^2$. This is exactly the observed scaling of Λ.

Other approaches use the entropy S or the energy density directly, and fail:

- Using S directly (as in Padmanabhan's [16] $N_{\mathrm{sur}} = N_{\mathrm{bulk}}$): the log term $\delta \ln R_H$ is subdominant to αA_H by a factor of $\ln(L_H)/A_H \sim 10^{-120}$, giving $\Lambda \propto \ln(L_H)/L_H^4$.

- Using the trace anomaly energy density: $\rho_{\text{vac}} \propto H^4 \propto 1/L_H^4$, again 120 orders too small.

The essential point is physical: the Clausius relation $-dE = T\,dS$ converts dS/dA into a contribution to the gravitational field equations. The $1/A$ term in dS/dA produces a $1/L_H^2$ scaling because $A_H = 4\pi L_H^2$. No other combination of entanglement entropy quantities achieves this scaling without introducing new physics.

7 The Case for $\Lambda_{\text{bare}} = 0$

The assumption $\Lambda_{\text{bare}} = 0$ is the single non-theorem in our derivation. We do not claim to prove it. But the case is stronger than a mere assertion. We develop it in three steps: a structural argument from Jacobson's framework, a quantitative demonstration that the vacuum energy is already encoded in α, and an exact identity in 1+1 dimensions that makes this encoding a theorem rather than an observation.

7.1 Completeness of the entropic framework

In Jacobson's derivation, Newton's constant $G = 1/(4\alpha)$ is determined entirely by the entanglement entropy. There is no "bare G" to which quantum corrections are added. The area law absorbs all of the gravitational coupling. If we accept this for G, consistency suggests the same for Λ: it too should be determined entirely by the entropy structure, with no additional bare parameter.

7.2 The vacuum energy is already in α

The vacuum energy that would normally contribute to Λ_{bare} in quantum field theory arises from the same quantum fluctuations that produce the entanglement entropy. We can demonstrate this explicitly on the lattice.

The entanglement entropy is computed from the restricted two-point correlation matrices $X_{ij} = \langle \phi_i \phi_j \rangle$ and $P_{ij} = \langle \pi_i \pi_j \rangle$, with i, j in the subregion V. In the vacuum state, these are

$$X_{ij} = \sum_{\mathbf{k}} \frac{f_{\mathbf{k}}(i)\, f_{\mathbf{k}}(j)}{2\,\omega_{\mathbf{k}}}, \qquad P_{ij} = \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}\, f_{\mathbf{k}}(i)\, f_{\mathbf{k}}(j)}{2}. \tag{25}$$

The vacuum energy density is

$$\rho_{\text{vac}} = \frac{1}{2V} \sum_{\mathbf{k}} \omega_{\mathbf{k}}. \tag{26}$$

Both α and ρ_{vac} are built from the same set of mode frequencies $\{\omega_{\mathbf{k}}\}$. Their UV structure is identical: $\alpha \sim \Lambda_{\text{UV}}^2$ and $\rho_{\text{vac}} \sim \Lambda_{\text{UV}}^4$, so on dimensional grounds alone the ratio $\alpha/\rho_{\text{vac}} \sim \Lambda_{\text{UV}}^{-2}$ should scale as the inverse cutoff squared. For a fixed lattice discretisation (fixed cutoff), the ratio should therefore be independent of the lattice size N.

We tested this directly. Table 3 shows α/ρ_{vac} across five lattice sizes.

The ratio is constant to 3.3%. We emphasise that this constancy is *expected* on dimensional grounds and is therefore a necessary but not sufficient condition for the double-counting argument. The physical content is that both quantities are determined by the same vacuum two-point function: ρ_{vac} sums the $\omega_{\mathbf{k}}$ globally, while α is determined by the restriction of the same $\omega_{\mathbf{k}}$-dependent correlators to a subregion boundary. The

N	α	ρ_{vac}	α/ρ_{vac}
10	0.0228	1.1977	0.01904
12	0.0213	1.1971	0.01782
14	0.0231	1.1966	0.01933
16	0.0231	1.1963	0.01929
18	0.0235	1.1960	0.01964
Mean ± std			$\mathbf{0.01902 \pm 0.00063}$
Coefficient of variation			**3.3%**

Table 3: The ratio α/ρ_{vac} across lattice sizes. Both α and ρ_{vac} individually vary with N, but their ratio is constant to 3.3%. The vacuum energy is encoded in α.

UV-divergent part of the entropy—the area law—and the vacuum energy are built from the same degrees of freedom.

The physical decomposition is therefore:

$$S = \underbrace{\alpha\, A}_{\text{UV-divergent, encodes } \rho_{\text{vac}}} + \underbrace{\delta \ln R}_{\text{UV-finite, determines } \Lambda} + \cdots \tag{27}$$

Adding $\Lambda_{\text{bare}} = 8\pi G\, \rho_{\text{vac}}$ on top of the entanglement contribution would count the vacuum energy twice: once through α (which determines G) and again through ρ_{vac} (which would determine Λ_{bare}).

7.3 An exact identity in 1+1 dimensions

The double-counting argument becomes a theorem in 1+1 dimensions, where the connection between the subleading entropy correction and the vacuum energy can be verified exactly.

Consider a massless scalar field on a periodic chain of N sites. The Casimir energy is

$$E_{\text{Casimir}} = \frac{1}{2}\sum_{k\neq 0} \omega_k - \frac{N}{2\pi}\int_0^{2\pi} \frac{\omega(k)}{2}\, dk = -\frac{\pi\, c}{6\, N}\,, \tag{28}$$

where $c = 1$ is the central charge. The entanglement entropy of a subregion of ℓ sites on a circle of N sites is

$$S(\ell) = \frac{c}{3}\ln\!\left(\frac{N}{\pi}\sin\frac{\pi\ell}{N}\right) + \gamma\,. \tag{29}$$

Both the subleading finite-size correction to S and the Casimir energy are manifestations of the same quantity: the trace anomaly. In 1+1 dimensions, $\langle T^a{}_a\rangle = c\,R/(24\pi)$, and

$$\rho_{\text{Casimir}} = \frac{E_{\text{Casimir}}}{N} = -\frac{\pi\, c}{6\, N^2}\,. \tag{30}$$

We verified this identity on the lattice to four decimal places:

The lesson is this: in 1+1 dimensions, the subleading correction to the entanglement entropy *is* the vacuum energy. They are not two independent quantities that must be added. They are two descriptions of the same physics—the trace anomaly—computed in different ways.

N	E_{Casimir} (lattice)	$-\pi/(6N)$ (CFT)	Ratio
50	-0.010473	-0.010472	1.0001
100	-0.005236	-0.005236	1.0000
200	-0.002618	-0.002618	1.0000
400	-0.001309	-0.001309	1.0000
800	-0.000654	-0.000654	1.0000

Table 4: Casimir energy of a massless periodic chain. The lattice result matches the CFT prediction $E_{\text{Casimir}} = -\pi/(6N)$ to better than 10^{-4}.

In 3+1 dimensions, the same structural pattern holds. The log correction $\delta \ln R$ is related to the four-dimensional trace anomaly $\langle T^a{}_a \rangle = a\, E_4 + c\, W^2$, where the coefficient δ is proportional to the a-anomaly coefficient [7, 8]. The area-law coefficient α encodes the UV-divergent vacuum energy (as Table 3 confirms), while the UV-finite log correction δ generates the cosmological constant through the horizon first law. Treating the vacuum energy as an independent source of Λ on top of this would double-count the same quantum correlations.

We emphasise that these arguments, taken together, are strong but not conclusive. The 1+1-dimensional identity is exact; the extension to 3+1 dimensions relies on the structural analogy between the two-dimensional trace anomaly and its four-dimensional counterpart. $\Lambda_{\text{bare}} = 0$ remains an assumption. If it is wrong, the formula (16) still gives the *entanglement contribution* to Λ, which is a well-defined and novel quantity in its own right.

8 Results and Discussion

8.1 Numerical prediction

Single scalar. With $\alpha = 0.019$ (global cutoff), $\delta = -1/90$ (confirmed numerically in Table 1), and $L_H = 8.8 \times 10^{60}\, l_{\text{Pl}}$:

$$\Lambda_{\text{scalar}} = \frac{1/90}{4 \times 0.019 \times (8.8 \times 10^{60})^2} = 1.9 \times 10^{-123}\,, \tag{31}$$

in Planck units. The observed value is $\Lambda_{\text{obs}} = 1.1 \times 10^{-122}$, giving

$$\frac{\Lambda_{\text{scalar}}}{\Lambda_{\text{obs}}} = 0.17\,. \tag{32}$$

The single-scalar prediction is a factor of 6 below observation—within an order of magnitude, with the correct $1/L_H^2$ scaling.

Standard Model. With the SM totals from (19)–(20):

$$\Lambda_{\text{SM}} = \frac{11.06}{4 \times 2.54 \times (8.8 \times 10^{60})^2} = 1.4 \times 10^{-122}\,, \tag{33}$$

giving

$$\frac{\Lambda_{\text{SM}}}{\Lambda_{\text{obs}}} = 1.3\,. \tag{34}$$

The Standard Model prediction overshoots observation by only 30%. This is a striking result: the same formula that gives a factor-of-6 undershoot for a single scalar produces agreement to within 30% when all Standard Model species are included, because vectors dominate the numerator (Section 5).

The connection between $\Lambda/\Lambda_{\rm obs}$ and the self-consistency ratio $|\delta|/(12\alpha)$ is as follows. In a pure de Sitter universe, $\Lambda/\Lambda_{\rm obs} = |\delta|/(12\alpha)$. For a single scalar this is 0.048; for the SM, 0.36. The reason the actual ratio (34) gives 1.3 rather than 0.36 is that $\Lambda_{\rm obs}$ is evaluated in the present matter-plus-Λ universe, where $H^2 = \Lambda/(3\,\Omega_\Lambda)$ with $\Omega_\Lambda \approx 0.68$, providing a factor $1/\Omega_\Lambda \approx 1.47$. The remaining factor of ~ 2.4 reflects the difference between the Hubble length $L_H = 1/H_0$ and the de Sitter radius $L_{\rm dS} = \sqrt{3/\Lambda}$.

The 30% agreement with observation for the Standard Model, while striking, is not independent of the self-consistency ratio. They are two manifestations of the same quantity $|\delta_{\rm SM}|/(12\alpha_{\rm SM}) = 0.36$, evaluated in different cosmological backgrounds. The self-consistency condition remains violated by a factor of ~ 3 (Section 4.3), so the precise coefficient is not yet fully under theoretical control—but the discrepancy is now at a level where non-equilibrium corrections to the horizon first law could plausibly close the gap.

8.2 Comparison: cube vs sphere

Table 5 compares the Λ predictions from the two entangling surface geometries.

Source	δ	α	$\Lambda/\Lambda_{\rm obs}$	$\lvert\delta\rvert/(12\alpha)$
Single scalar (sphere)	$-1/90$	0.019	0.17	0.048
Standard Model	**−11.06**	**2.54**	**1.3**	**0.36**
Cube ($N = 48$, Dirichlet)	-0.076	0.024	0.93	0.264

Table 5: Predictions from different field contents and entangling surface geometries. The Standard Model prediction uses spherical δ values from the trace anomaly and lattice α values with global cutoff (Section 4.3). The cube is contaminated by edge/corner terms (Section 2.4) and is shown for comparison only.

The single-scalar sphere result uses the analytically predicted trace anomaly coefficient, confirmed here to 1% accuracy with no geometric contamination. The Standard Model result combines analytically known δ values for all species with lattice α values measured with consistent methodology.

The cubic value $|\delta_{\rm cube}|/(12\alpha) = 0.264$ is closer to the de Sitter self-consistency condition (17) than the single-scalar sphere value 0.048, but this is an artifact: the edge and corner contamination in $\delta_{\rm cube}$ inflates $|\delta|$ by approximately $7\times$ relative to the universal value. The Standard Model value of 0.36, by contrast, arises from the physical content of the theory—the dominance of vector boson anomaly coefficients (Section 5).

8.3 The 1D Casimir template

The exact identity between the subleading entropy correction and the Casimir energy in 1+1 dimensions—established in Section 7.3 and verified to four decimal places in Table 4—provides the template for the 3+1-dimensional mechanism. In both cases, the trace anomaly generates a UV-finite correction to the entropy that is physically identical to the vacuum energy. The only difference is the mechanism by which it enters the field equations: in 1+1 dimensions through the central charge, in 3+1 dimensions through the cosmological horizon first law.

8.4 Comparison with the literature

Several approaches obtain the scaling $\Lambda \propto 1/L_H^2$. We compare our framework against the most relevant.

Cohen-Kaplan-Nelson (CKN) [15]. CKN argued that the Bekenstein entropy bound implies a UV-IR connection in effective field theory, giving $\rho_\Lambda \lesssim M_{\rm Pl}^2/L^2$ where L is the infrared cutoff. This yields the correct $1/L_H^2$ scaling but provides an *inequality*, not a prediction—the coefficient is undetermined. Li [20] promoted this to the holographic dark energy model $\rho_{\rm de} = 3c^2 M_{\rm Pl}^2/L^2$ with c as a free parameter fit to data ($c \approx 0.7$). Our framework determines the coefficient from QFT—it is $|\delta|/(4\alpha)$. For a single scalar, this gives a value a factor of 6 below observation; for the full Standard Model, it overshoots by only 30%. The advantage is that the coefficient is *computed*, not fitted.

Padmanabhan [16]. Related Λ to cosmic information content ($N_{\rm sur} - N_{\rm bulk}$), obtaining the same scaling from a different framework.

Verlinde [17]. Argued that gravity is an entropic force but did not address the cosmological constant.

Solodukhin [6]. Connected entanglement entropy to the cosmological constant through induced gravity, using the UV-divergent area term rather than the UV-finite log correction.

We note that the comparison against the naïve vacuum energy estimate ($\rho_{\rm vac} \sim M_{\rm Pl}^4$, off by 10^{122}) overstates the achievement, since few physicists take the unrenormalised estimate seriously. The fairer comparison is against other $1/L_H^2$ approaches, where our contribution is a specific, computable coefficient derived from a lattice calculation, with δ confirmed to 1% against the analytically known trace anomaly. For the Standard Model, the coefficient is within 30% of the observed value, though the self-consistency condition is not yet exactly satisfied (Section 4.3). The mechanism is distinct and testable.

8.5 Limitations

We are honest about what this work does not achieve.

Self-consistency gap. The de Sitter self-consistency condition $|\delta|/(12\alpha) = 1$ (Section 4.3) is violated by a factor of ~ 3 for the Standard Model ($R_{\rm SM} = 0.36$) and a factor of ~ 21 for a single scalar ($R = 0.048$). For a single species the failure is species-independent, but the Standard Model's mixed field content breaks this independence and substantially reduces the gap. The remaining factor of ~ 3 is at a level where non-equilibrium corrections to the horizon first law [18, 19] could plausibly account for the discrepancy, but this has not been demonstrated. Until it is resolved, the precise numerical coefficient in the Λ prediction is not fully under theoretical control, even though the scaling $\Lambda \propto 1/L_H^2$ and the order of magnitude are correct.

Free fields only. The lattice computation uses free fields (scalars, Dirac fermions, and vectors). The entanglement entropy of interacting fields has the same general form (2), and the trace anomaly coefficients δ are analytically known for all Standard Model species. The area-law coefficients α are computed on the lattice; their ratios ($\alpha_{\rm Dirac}/\alpha_{\rm s} = 4.6$, matching heat kernel) suggest the lattice values are reliable, but interactions could modify α at higher order.

Cai-Kim framework. The horizon first law at the cosmological horizon is a well-established framework used by many authors, but it is a physical framework, not a mathematical theorem. It assumes that thermodynamic relations hold at the apparent horizon with the full log-corrected entropy.

Finally, $\Lambda_{\text{bare}} = 0$ is an assumption. If a nonzero bare cosmological constant exists, the formula (16) gives only the entanglement contribution, and the full Λ would require the (unknown) bare value as well.

9 Conclusion

We have presented a derivation of the cosmological constant from the entanglement entropy of quantum fields. The logic is:

1. The entanglement entropy has the form $S = \alpha A + \delta \ln R + \cdots$, where δ is UV-finite and R is the radius of the entangling surface. (QFT theorem.)

2. Jacobson's argument derives Einstein's equations from the area law, leaving Λ undetermined. (GR theorem.)

3. The log correction is invisible at local Rindler horizons ($A \to \infty$) but visible at the cosmological horizon (A_H finite).

4. The Cai-Kim horizon first law at the cosmological horizon generates $\Lambda = |\delta|/(4\alpha L_H^2)$.

5. Under the assumption $\Lambda_{\text{bare}} = 0$, with the Standard Model field content ($\delta_{\text{SM}} = -11.06$, $\alpha_{\text{SM}} = 2.54$), this gives $\Lambda_{\text{SM}}/\Lambda_{\text{obs}} \approx 1.3$—within 30% of observation.

The value $\delta = -1/90$ per scalar is an analytically known universal quantity—the type-A trace anomaly coefficient—confirmed numerically to 1% by the angular momentum decomposition. The trace anomaly coefficients for fermions ($\delta = -11/360$ per Weyl spinor) and vectors ($\delta = -31/45$) are likewise analytically known. The lattice area-law coefficients α have been measured for all three species with consistent methodology; the Dirac-to-scalar ratio of 4.6 agrees with the heat kernel prediction of 4.

A key finding is that species independence—which holds exactly for identical fields—breaks for the mixed field content of the Standard Model. The ratio $|\delta|/\alpha$ is 31× larger for vectors than for scalars, so vectors dominate the Standard Model prediction. This breaking improves the prediction by a factor of 7.5 relative to the single-scalar result.

The de Sitter self-consistency condition $|\delta|/(12\alpha) = 1$ is violated by a factor of ~ 3 for the Standard Model ($R_{\text{SM}} = 0.36$). This is a dramatic improvement over the single-scalar failure ($R = 0.048$, a factor of 21). The remaining discrepancy is at a level where non-equilibrium corrections to the horizon first law could plausibly close the gap: Eling, Guedens, and Jacobson [18] showed that entropy corrections beyond the area law generate internal entropy production terms ($d_i S$) that the equilibrium Clausius relation omits, and Chirco and Liberati [19] showed these are related to gravitational dissipation at the horizon. A dedicated calculation of $d_i S$ for the log-corrected entropy at a quasi-de-Sitter horizon is the most concrete path forward.

If the remaining factor of ~ 3 can be resolved, the cosmological constant would not be a free parameter of nature but a consequence of the entanglement structure of the Standard Model quantum fields.

A Lattice Methods

A.1 Cubic lattice

We work on a three-dimensional cubic lattice of N^3 sites with lattice spacing $a = 1$. The free massless scalar field has Hamiltonian

$$H = \frac{1}{2}\sum_i \pi_i^2 + \frac{1}{2}\sum_{\langle ij\rangle}(\phi_i - \phi_j)^2 , \tag{35}$$

where the sum $\langle ij\rangle$ runs over nearest neighbours.

Mode decomposition. With Dirichlet boundary conditions, the mode functions are $f_{\mathbf{k}}(\mathbf{x}) = (2/(N+1))^{3/2}\prod_{d=1}^{3}\sin(\pi k_d x_d/(N+1))$, with frequencies $\omega_{\mathbf{k}}^2 = \sum_{d=1}^{3} 2(1-\cos(\pi k_d/(N+1)))$. The two-point functions in the vacuum are:

$$X_{ij} = \sum_{\mathbf{k}} \frac{f_{\mathbf{k}}(i)\, f_{\mathbf{k}}(j)}{2\omega_{\mathbf{k}}} , \qquad P_{ij} = \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}\, f_{\mathbf{k}}(i)\, f_{\mathbf{k}}(j)}{2} . \tag{36}$$

Subregion entropy. For a cubic subregion of side L, we restrict X and P to the L^3 sites inside the cube. The symplectic eigenvalues are obtained from $\nu_k = \sqrt{\lambda_k(X_{\text{sub}}\, P_{\text{sub}})}$, and the entropy follows from (4).

Null-space extraction of δ. The design matrix for the four-parameter fit (8) has columns $(6L^2, 12L, \ln L, 1)$ for each L value. We compute a vector $\mathbf{c}$ in the left null space of the first, second, and fourth columns (area, perimeter, constant), so that $\mathbf{c}^T\mathbf{S} = \delta\cdot\mathbf{c}^T \ln\mathbf{L}$. This gives $\delta = \mathbf{c}^T\mathbf{S}/(\mathbf{c}^T \ln\mathbf{L})$ without fitting α, β, or γ.

Lattice sizes $N = 22$, 28, 36, and 48 were used, with subregion sizes $L = 2, \ldots, N/2-1$.

A.2 Spherical decomposition

Following Lohmayer et al. [10], the free scalar field in 3+1 dimensions is expanded in spherical harmonics. Each (l, m) sector reduces to a 1D radial chain with N_{radial} sites at positions $r_j = j\,a$ ($j = 1, \ldots, N_{\text{radial}}$; $a = 1$). Using canonical variables $q_j = j\,\phi_j$ and $p_j = \pi_j/j$, the coupling matrix is tridiagonal:

$$K_l'[j,j] = \frac{(j-\frac{1}{2})^2 + (j+\frac{1}{2})^2 + l(l+1)}{j^2} + m^2 , \tag{37}$$

$$K_l'[j,j{+}1] = -\frac{(j+\frac{1}{2})^2}{j\,(j+1)} , \tag{38}$$

with Dirichlet boundary conditions ($\phi_0 = \phi_{N+1} = 0$).

Diagonalisation. We use `scipy.linalg.eigh_tridiagonal` to obtain eigenvalues ω_k^2 and eigenvectors V_{jk} in $O(N_{\text{radial}}^2)$ time per channel.

Covariance matrices. For the subregion $[1, \ldots, n]$ (a sphere of radius n):

$$X_{jk} = \frac{1}{2}\sum_m \frac{V_{jm}\, V_{km}}{\omega_m} , \qquad P_{jk} = \frac{1}{2}\sum_m V_{jm}\, V_{km}\, \omega_m , \tag{39}$$

restricted to $j, k \le n$. Symplectic eigenvalues and entropy follow from (4).

Summation over channels. The total entropy is $S_{\text{total}}(n) = \sum_{l=0}^{l_{\max}} (2l+1)\, S_l(n)$, with $l_{\max} = 10n$ (proportional cutoff). To avoid storing all eigenvector matrices simultaneously, we process one l-channel at a time, accumulating $S(n)$ incrementally. Memory cost is $O(N_{\text{radial}}^2)$ regardless of $l_{\max}$.

Third-difference extraction. We compute $S_{\text{total}}(n)$ for consecutive $n = n_{\min}-1, \ldots, n_{\max}+2$ and form $\Delta^3 S(n)$ as in (7). A two-parameter fit $\Delta^3 S = A/n^3 + B/n^4$ gives $\delta = A/2$. At $N_{\text{radial}} = 1000$ with $n \in [25, 100]$, this achieves 1% accuracy.

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